The theory of the exponential differential equations of semiabelian varieties

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Abstract

The complete first order theories of the exponential differential equations of semiabelian varieties are given. It is shown that these theories also arise from an amalgamation-with-predimension construction in the style of Hrushovski. The theories include necessary and sufficient conditions for a system of equations to have a solution. The necessary conditions generalize Ax's differential fields version of Schanuel's conjecture to semiabelian varieties. There is a purely algebraic corollary, the "Weak CIT" for semiabelian varieties, which concerns the intersections of algebraic subgroups with algebraic varieties.

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1 Introduction

1.1 The exponential differential equation

Let $\langle F; +, \cdot, D\rangle$ be a differential field of characteristic zero, and consider the *exponential differential equation* $Dx = \frac{Dy}{y}$. If F is a field of meromorphic functions in a variable t, with D being $\frac{d}{dt}$, then this is the differential equation satisfied by any $x(t), y(t) \in F$ such that $y(t) = e^{x(t)}$.

James Ax proved the following differential fields version of Schanuel's conjecture.

Theorem 1.1 (Ax, [Ax71]). *Let F be a field of characteristic zero, D be a derivation on F and C be the constant subfield. Suppose $n \geqslant 1$ and $x_1, y_1, \ldots, x_n, y_n \in F$ are such that $Dx_i = \frac{Dy_i}{y_i}$ for each i, and the Dx_i are $\mathbb{Q}$-linearly independent. Then* $\operatorname{td}(x_1, y_1, \ldots, x_n, y_n/C) \geqslant n + 1$.

Here and throughout this paper, $\mathrm{td}(X/C)$ means the transcendence degree of the field extension $C(X)/C$.

The theorem can be viewed as giving a restriction on the systems of equations which have solutions in a differential field. In this paper it is shown that Ax's theorem is the only restriction on a system of instances of the exponential differential equation and polynomial equations having solutions in a differential field. This is done by proving a matching *existential closedness* theorem, stating that certain systems of equations do have solutions when the field F is differentially closed. The theorem says roughly that certain algebraic subvarieties $V \subseteq F^{2n}$, whose images under projections are suitably large and which we call *rotund subvarieties*, must have nonempty intersection with the set of n-tuples of solutions of the exponential differential equation.

1.2 Semiabelian varieties

We consider not just the exponential differential equation given above, but also the exponential differential equations of every semiabelian variety defined over the field of constants. Next we explain what these equations are, and the language we use to study them. The foundations of algebraic geometry we use are those standard in model theory, which can be found in [Pil98] or in [Mar00]. In particular, we generally work with a fixed algebraically closed field F (always of characteristic zero in this paper), and identify a variety over F with its set of F-points, although we may write the latter also as $V(F)$. If K is another field with $K \subseteq F$ or $F \subseteq K$ then we write $V(K)$ for the K-points of V.

In characteristic zero, we can define a *semiabelian variety* S to be a connected, commutative algebraic group, with no algebraic subgroup isomorphic to the additive group $\mathbb{G}_{\mathrm{a}}$. By Chevalley's theorem [Ser88, p40], every algebraic group G can be given as an extension

$$0 \to L \to G \to A \to 0$$

in a unique way, where A is an abelian variety (a connected projective algebraic group) and L is a linear group. If G is connected and commutative and the characteristic is zero then L is of the form ${\mathbb{G}_{\mathrm{a}}}^{l} \times {\mathbb{G}_{\mathrm{m}}}^{k}$ for some natural numbers l and k [Ser88, p40, p171]. So G is a semiabelian variety when, in addition, $l = 0$. Special cases include the multiplicative group $\mathbb{G}_{\mathrm{m}}$, its powers which are called algebraic tori, and elliptic curves which are one-

dimensional abelian varieties. Algebraic groups isomorphic to ${\mathbb{G}_{\mathrm{a}}}^n$ for some natural number n are called *vector groups*.

We need the notions of tangent bundles and the logarithmic derivative map. A good exposition is given in [Mar00], so we just summarize the essential properties we need. Given any connected commutative algebraic group G, its tangent bundle TG is also a connected commutative algebraic group. We write LG for the tangent space at the identity of G. (The L here stands for Lie algebra, but since the group is commutative, the Lie bracket is trivial.) LG is a vector group with $\dim LG = \dim G$, and TG is canonically isomorphic to $LG \times G$ as an algebraic group.

For any differential field $\langle F; +, \cdot, D\rangle$ and any commutative algebraic group G defined over the subfield of constants C, there is a *logarithmic derivative map*, which is a group homomorphism $lD_G : G(F) \to LG(F)$.

If G is a vector group then LG is canonically isomorphic to G. In particular, for any G, LLG is canonically isomorphic to LG. We have $lD_{\mathbb{G}_{\mathrm{a}}}(x) = Dx$ and, identifying $L\mathbb{G}_{\mathrm{m}}$ with $\mathbb{G}_{\mathrm{a}}$ we have $lD_{\mathbb{G}_{\mathrm{m}}}(y) = \frac{Dy}{y}$, so the usual exponential differential equation can be written as

$$lD_{L\mathbb{G}_{\mathrm{m}}}(x) = lD_{\mathbb{G}_{\mathrm{m}}}(y).$$

For a general semiabelian variety S, defined over the field of constants C, we define the *exponential differential equation of S* to be

$$lD_{LS}(x) = lD_S(y)$$

under the canonical identification of LLS and LS. The equation defines a differential subvariety of TS, which we denote by Γ_S. That is,

$$\Gamma_S = \left\{(x, y) \in LS \times S \mid lD_{LS}(x) = lD_S(y)\right\}.$$

As mentioned earlier, the usual exponential map satisfies the exponential differential equation of $\mathbb{G}_{\mathrm{m}}$. If S is a complex semiabelian variety, we may consider $S(\mathbb{C})$ as a complex Lie group, and $LS(\mathbb{C})$ can be identified with its universal covering space. The analytic covering map

$$LS(\mathbb{C}) \xrightarrow{\exp_S} S(\mathbb{C})$$

is called the *exponential map* of $S(\mathbb{C})$, and it can be shown via a Lie theory argument that this map satisfies the exponential differential equation for S. This is one motivation for considering these equations.

Having explained the equations under consideration, we now explain the context in which we study them. Let $\langle F; +, \cdot, C, D\rangle$ be a differential field of characteristic 0, with C being the constant subfield. Let C_0 be a countable subfield of C, and let $\mathcal{S}$ be a collection of semiabelian varieties, each defined over C_0. Expand F by adding a symbol for Γ_S for each $S \in \mathcal{S}$ (of appropriate arity to be interpreted as a subset of TS) and by adding constant symbols for each element of C_0. Then forget the deriviation – consider the reduct $\langle F; +, \cdot, C, (\Gamma_S)_{S\in\mathcal{S}}, (\hat{c})_{c\in C_0}\rangle$. We call this language $\mathcal{L}_{\mathcal{S}}$.

We will give the complete first-order theory of this reduct, in the case where $\langle F; +, \cdot, D\rangle$ is a differentially closed field.

1.3 Outline of the paper

In section 2 of the paper we take the analogues for semiabelian varieties of Ax's theorem (see below) as a starting point. We observe that they can be seen as stating the positivity of a *predimension function*, as used by Hrushovski [Hru93] to construct his new strongly minimal theories – theories where there is a particularly simple and powerful dimension theory. This is easiest to see in the original multiplicative group setting. Write x, y for the tuples $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ in theorem 1.1, and define

$$\delta(x, y) = \mathrm{td}(x, y/C) - \mathrm{ldim}_{\mathbb{Q}}(x/C)$$

where the second term is the $\mathbb{Q}$-linear dimension of the span of the images of the x_i in the quotient $\mathbb{Q}$-vector space F/C. Then Ax's theorem is equivalent to the statement that for all tuples $x, y \in F^n$ satisfying the exponential differential equation, either $\delta(x, y) \geqslant 1$ or all the x_i and y_i lie in C.

Using this predimension function, and its generalisations, we construct abstract $\mathcal{L}_{\mathcal{S}}$-theories $T_{\mathcal{S}}$ via a category-theoretic version of Hrushovski's *amalgamation with predimension* technique. In particular, we obtain a pregeometry with its associated notion of dimension, and the definition (see 2.26) of the rotund subvarieties of the tangent bundles TS, which are those occuring in the existential closedness statements. However, we cannot at this stage of the paper show that $T_{\mathcal{S}}$ is first-order axiomatizable.

Section 3 starts by connecting the logarithmic derivatives with differential forms, and goes on to prove that the analogues of Ax's theorem (which we call *Schanuel properties*) do indeed hold in all differential fields. As in Ax's paper, we prove a statement for many commuting derivations. The simpler statement for just one derivation is as follows.

Theorem (3.8, the Schanuel Property, one derivation version)**.** *Let F be a differential field of characteristic zero, with constant subfield C. Let S be a semiabelian variety defined over C, of dimension n.*

Suppose that $(x, y) \in \Gamma_S$ and $\operatorname{td}(x, y/C) < n + 1$. Then there is a proper algebraic subgroup H of S and a constant point γ of TS such that (x, y) lies in the coset $\gamma \cdot TH$.

We also prove that the existential closedness axioms hold in differentially closed fields.

Theorem (3.10, Existential Closedness)**.** *Let F be a differentially closed field of characteristic zero, and S a semiabelian variety defined over C. Then for each irreducible rotund subvariety V of TS, and each parametric family $(W_e)_{e\in Q(C)}$ of proper subvarieties of V, with Q a constructible set defined over C, there is $g \in \Gamma_S \cap V \smallsetminus \bigcup_{e\in Q(C)} W_e$.*

This theorem extends work of Crampin [Cra06], who considered a case where the variety V is defined over the constant subfield, just for the multiplicative group.

In section 4 we apply the compactness theorem of first-order logic to the Schanuel properties proved in section 3 to show that they are first-order expressible, and to deduce a result in diophantine geometry, concerning the intersections of algebraic subgroups of semiabelian varieties with algebraic varieties.

Theorem (4.6, "Weak CIT" for semiabelian varieties)**.** *Let S be a semiabelian variety defined over an algebraically closed field C of characteristic zero. Let $(U_p)_{p\in P}$ be a parametric family of algebraic subvarieties of S. There is a finite family $\mathcal{J}_U$ of proper algebraic subgroups of S such that, for any coset $\kappa = a \cdot H$ of any algebraic subgroup H of S and any $p \in P(C)$, if X is an irreducible component of $U_p \cap \kappa$ and*

$$\dim X = (\dim U_p + \dim \kappa - \dim S) + t$$

with $t > 0$, an atypical component of the intersection, then there is $J \in \mathcal{J}_U$ of codimension at least t and $s \in S(C)$ such that $X \subseteq s \cdot J$.

This is a weak version of the *Conjecture on the intersection of algebraic subgroups with subvarieties* stated by Zilber in [Zil02], and is the natural generalization to semiabelian varieties of the version proved there for algebraic

tori. ("CIT" stands for the Conjecture on Intersections with Tori.) For a discussion of results and conjectures of this form, see [Zil02] and [BMZ07].

We then use this weak CIT result to show that the notion of rotundity of a subvariety is definable, and hence that the existential closedness property is first-order expressible. Thus the theories $T_{\mathcal{S}}$ are first-order, and we then show they are complete. Finally, we give two simple model-theoretic properties of the $T_{\mathcal{S}}$.

I believe the results of this paper can be generalised to arbitrary commutative algebraic groups, although vector groups must be treated separately because their exponential maps are just the identity maps. Indeed, Bertrand [Ber08] has proved the Schanuel property for commutative algebraic groups with no vectorial quotients, a generalization of semiabelian varieties. He makes use of another paper of Ax [Ax72], and considers only the case where the differential field is a field of meromorphic functions. The method of [Kir05] and of §5.5 of [Kir06] generalizes Bertrand's result to any differential field. In these cases, the groups are still defined over the constant field C (or, essentially equivalently, are isoconstant). Bertrand and Pillay have also considered Schanuel properties in the non-isoconstant case [BP08].

Much of the work of this paper was done as part of my DPhil thesis [Kir06] under the supervision of Boris Zilber, and his great influence will be clear to anyone who knows his work.

2 Amalgamation

In this section we put aside differential fields and construct an abstract $\mathcal{L}_{\mathcal{S}}$-structure and its theory $T_{\mathcal{S}}$. In section 3 we show that the reducts of differentially closed fields are models of $T_{\mathcal{S}}$. It is not immediate that $T_{\mathcal{S}}$ is first-order axiomatizable, but this is proven in section 4. We start by giving the universal part of $T_{\mathcal{S}}$.

2.1 The universal theory

Fix a countable field C_0 of characteristic zero, and a collection $\mathcal{S}$ of semiabelian varieties, each defined over C_0. We assume also that C_0 is large enough that every algebraic homomorphism between any members of $\mathcal{S}$ is defined over C_0. For example, if $\mathcal{S}$ is the collection $\{{\mathbb{G}_\mathrm{m}}^n \mid n \in \mathbb{N}\}$ of algebraic tori, then we can just take $C_0 = \mathbb{Q}$. In any case it suffices to take C_0 to be al-

gebraically closed. Recall that the language $\mathcal{L}_{\mathcal{S}}$ is $\langle +, \cdot, C, (\Gamma_S)_{S\in\mathcal{S}}, (\hat{c})_{c\in C_0}\rangle$, the field language augmented by relation symbols for the constant field and for each solution set Γ_S, and by constant symbols for the elements of C_0. The theory $T^U_{\mathcal{S}}$ is given as follows.

- U1 F is an algebraically closed field, C is a (relatively) algebraically closed subfield, and the subfield C_0 of C is named by parameters.
- U2 For each $S \in \mathcal{S}$, Γ_S is a subgroup of TS.
- U3 For each $S \in \mathcal{S}$, $TS(C) \subseteq \Gamma_S$
- U4 $(0, y) \in \Gamma_S \iff y \in S(C)$ and $(x, 1) \in \Gamma_S \iff x \in LS(C)$, where 0 is the identity of LS and 1 is the identity of S.
- U5 If $S_1 \xrightarrow{f} S_2$ is an algebraic group homomorphism then $(Tf)(\Gamma_{S_1}) \subseteq \Gamma_{S_2}$, and if f is an isogeny then also $\Gamma_{S_1} = (Tf)^{-1}(\Gamma_{S_2})$.
- U6 For each $S_1, S_2 \in \mathcal{S}$, if $S_1 \subseteq S_2$ then $\Gamma_{S_1} = \Gamma_{S_2} \cap TS_1$.
- U7 For each $S_1, S_2 \in \mathcal{S}$, $\Gamma_{S_1\times S_2} = \Gamma_{S_1} \times \Gamma_{S_2}$.
- SP For each $S \in \mathcal{S}$, if $g \in \Gamma_S$ and $\operatorname{td}(g/C) < \dim S + 1$ then there is a proper algebraic subgroup H of S and $\gamma \in TS(C)$ such that g lies in the coset $\gamma \cdot TH$.

Lemma 2.1. *The axioms U1 — U7 can all be expressed as first order axiom schemes in the language $\mathcal{L}_{\mathcal{S}}$.*

Proof. This is almost immediate. For U5, recall that by assumption on C_0, every algebraic homomorphism $S_1 \xrightarrow{f} S_2$ is defined over C_0, and hence is $\emptyset$-definable in $\mathcal{L}_{\mathcal{S}}$. □

The last axiom, SP, is the Schanuel property. Since each $S \in \mathcal{S}$ has only countably many proper algebraic subgroups and there are only countably many polynomials, it follows that SP can be expressed as a sentence in the infinitary language $\mathcal{L}_{\omega_1,\omega}$. We show later (corollary 4.4) that SP can also be expressed as a first order axiom scheme.

The superscript "U" in $T^U_{\mathcal{S}}$ stands for universal. The theory is universal, that is, if M is a model and N is a substructure of M then N is also a model, with the exception of the part of U1 that says that the field F is

algebraically closed. It will be convenient to work in a setting in which we only consider substructures whose underlying field is algebraically closed. In this non-elementary setting, the theory $T^U_{\mathcal{S}}$ is precisely the "theory of substructures".

If $\mathcal{S}$ is not closed under products, then for $S_1, S_2 \in \mathcal{S}$ we can use axiom U7 to define $\Gamma_{S_1 \times S_2}$. Thus we may assume that $\mathcal{S}$ is closed under products. Similarly, using U6 we may assume that $\mathcal{S}$ is closed under taking (connected) subgroups, and using U5 we may assume that $\mathcal{S}$ is closed under quotients. An *isogeny* is a surjective homomorphism with finite kernel. Groups S_1 and S_2 are said to be *isogenous* iff there is S_3 and isogenies $S_3 \longrightarrow S_1$ and $S_3 \longrightarrow S_2$. By U5 we can also assume that $\mathcal{S}$ is closed under the equivalence relation of isogeny.

2.2 The category $\mathcal{K}$

We now use Hrushovski's amalgamation-with-predimension technique to produce a "countable universal domain", U, for $T^U_{\mathcal{S}}$. From the construction of U we will obtain an axiomatization of its complete theory, $T_{\mathcal{S}}$. Again, it will be clear that the axiomatization is expressible in $L_{\omega_1\omega}$. We will later extract the first order part of the theory.

We apply the amalgamation construction not to the category of all countable models of $T^U_{\mathcal{S}}$, but to a subcategory. Fix a countable algebraically closed field C of characteristic zero, containing C_0. Unless otherwise noted, we take C to have a transcendence degree $\aleph_0$ over C_0.

Take $\mathcal{K}$ to be the category of models of the theory $T^U_{\mathcal{S}}$ which have this given field C, with arrows being embeddings of $\mathcal{L}_{\mathcal{S}}$-structures which fix C. Because we are working in a more abstract setting than usual, the following lemma actually requires a proof.

Lemma 2.2. *The category $\mathcal{K}$ has intersections, that is, for each $B \in \mathcal{K}$, and each family $(A_i \hookrightarrow B)_{i \in I}$ of substructures of B, there is a limit $\bigcap_{i \in I} A_i \hookrightarrow B$ of the obvious diagram this defines. Furthermore the underlying field of this intersection is simply the intersection of the underlying fields of the substructures.*

Proof. The axiomatization of $T^U_{\mathcal{S}}$ is universal, apart from the axiom scheme which says that the field is algebraically closed. The intersection of algebraically closed subfields of a field is algebraically closed, and any substructure of a model of a universal theory is also a model of that theory, so the

category of models of $T_{\mathcal{S}}^U$ has intersections. The intersection of extensions of C is also an extension of C. $\square$

Using this lemma, if $B \in \mathcal{K}$ and X is a subset of B, we can define the substructure of B generated by X as $\langle X \rangle = \bigcap \{A \hookrightarrow B \mid X \subseteq A\}$, where $A \hookrightarrow B$ means that A is a subobject of B in $\mathcal{K}$. Note that $\langle X \rangle$ depends on B.

We say that B is *finitely generated* iff there is a finite subset X of B such that $B = \langle X \rangle$. In fact, for any $A \in \mathcal{K}$ and subset X of A, $\langle X \rangle$ is simply the algebraic closure of $C \cup X$ in A, so an object A of $\mathcal{K}$ is finitely generated iff $\mathrm{td}(A/C)$ is finite. Thus being a finitely generated object of $\mathcal{K}$ is not the same as being finitely generated as an $\mathcal{L}_{\mathcal{S}}$-structure. Indeed no objects of $\mathcal{K}$ are finitely generated as $\mathcal{L}_{\mathcal{S}}$-structures since they are all algebraically closed fields.

We write $A \subseteq_{f.g.} B$ to mean that A is a finitely generated substructure of B. From the above characterization it follows that any substructure of a finitely generated structure in $\mathcal{K}$ is also finitely generated.

2.3 The predimension function

The Schanuel property allows us to define a predimension function, δ, on the finitely generated objects of $\mathcal{K}$. It is defined in terms of transcendence degree and a *group rank*, which we define using the next series of lemmas.

Lemma 2.3. *If S_1 and S_2 are isogenous then Γ_{S_1} determines Γ_{S_2}.*

Proof. By the definition of isogeny, there are an S_3 and isogenies $f_1 : S_3 \to S_1$ and $f_2 : S_3 \to S_2$. By axiom U5, $\Gamma_{S_2} = (Tf_2)(Tf_1)^{-1}(\Gamma_{S_1})$. $\square$

Lemma 2.4. *For any extension $A \hookrightarrow B$ in $\mathcal{K}$ with B finitely generated, there is $S \in \mathcal{S}$ of maximal dimension such that there is $g \in \Gamma_S(B)$, not lying in an A-coset of TH for any proper algebraic subgroup H of S. Furthermore, this maximal S is uniquely defined up to isogeny, and determines Γ on B as follows.*

If $g' \in \Gamma_{S'}(B)$ for any $S' \in \mathcal{S}$, then there is S'' isogenous to S, $g'' \in \Gamma_{S''}(B)$, a homomorphism $S'' \xrightarrow{q} S'$, and $\gamma \in \Gamma_{S'}(A)$, such that $g' = (Tq)(g'') \cdot \gamma$, where $\cdot$ is the group operation in S'.

Proof. If $g \in \Gamma_S(B)$ and does not lie in an A-coset of TH for any proper algebraic subgroup H of S, then it does not lie in a C-coset and by the

Schanuel property SP, $\dim S < \mathrm{td}(g/C)$ or $\dim S = 0$. Also $\mathrm{td}(g/C) \leqslant \mathrm{td}(B/C)$, so the dimension of S is bounded. At least one such S exists (the zero-dimensional group), and hence a maximal such S exists.

Now let S be of maximal dimension and $g \in \Gamma_S(B)$ as described. Suppose $g' \in \Gamma_{S'}(B)$ for some $S' \in \mathcal{S}$. Then $(g, g') \in \Gamma_{S\times S'}(B) \subseteq T(S \times S')(B)$. By maximality of $\dim S$, there is an algebraic subgroup S'' of $S \times S'$, with $\dim S'' \leqslant \dim S$, such that (g, g') lies in an A-coset of TS''. Let $(\alpha, \beta) \in \Gamma_{S\times S'}(A)$ and $g'' \in \Gamma_{S''}(B)$ such that $(g, g') = g'' \cdot (\alpha, \beta)$. The projection maps

$$\begin{tikzcd} & S \times S' \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ S & & S' \end{tikzcd} \quad \text{restrict to} \quad \begin{tikzcd} & S'' \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S & & S' \end{tikzcd}$$

and we also have the maps Tp, Tq on the tangent bundles. Then $(Tp)(g'') = g \cdot (Tp)(\alpha)$, which lies in $T(p(S''))$, where $p(S'')$ is an algebraic subgroup of S. Now g does not lie in TH for any proper algebraic subgroup H of S, so $p(S'') = S$. Hence $\dim S'' = \dim S$ and p is an isogeny. Let $\gamma = (T\,\mathrm{pr}_2)(\beta)$. Then $g' = (Tq)(g'') \cdot \gamma$, where $g'' \in \Gamma_{S''}(B)$ and $\gamma \in \Gamma_{S'}(A)$ as required.

If $\dim S' = \dim S$ then the same argument shows that q is an isogeny. Hence S is unique up to isogeny. $\square$

Definition 2.5. For an extension $A \hookrightarrow B$ in $\mathcal{K}$, with B finitely generated, define $S^{\max}(B/A)$ to be a maximal $S \in \mathcal{S}$ such that there is $g \in \Gamma_S(B)$, not lying in an A-coset of TH for any proper algebraic subgroup H of S. A point $g \in \Gamma_{S^{\max}(B/A)}$ which witnesses the maximality is said to be a *basis for* $\Gamma(B/A)$. For a finitely generated $A \in \mathcal{K}$, define $S^{\max}(A) = S^{\max}(A/C)$.

Note that $S^{\max}(B/A)$ is defined only up to isogeny.

Proposition 2.6. *Let $A, B \in \mathcal{K}$ be finitely generated, with B an extension of A, that is, $A \subseteq B$. Then $S^{\max}(B)$ is an extension of $S^{\max}(A)$ in the group theory sense, that is, $S^{\max}(A)$ is a quotient of $S^{\max}(B)$. Furthermore, $S^{\max}(B/A)$ is the kernel of the quotient map.*

Proof. Let $b \in \Gamma_{S^{\max}(B)}$ and $a \in \Gamma_{S^{\max}(A)}$ be bases, and write S_B for $S^{\max}(B)$ and S_A for $S^{\max}(A)$. Then, replacing S_B by an isogenous group if necessary, there is a quotient map $S_B \xrightarrow{q} S_A$, and $\gamma \in S_A(C)$ such that $a = (Tq)(b) \cdot \gamma$. Thus $(Tq)(b) \in S_A(A)$, so b lies in an A-coset of TH, where H is the kernel of q. Say $b = e \cdot \alpha$, with $e \in H(B)$ and $\alpha \in S_B(A)$.

We will show that e is a basis for $\Gamma(B/A)$. Firstly, e does not lie in an A-coset of TJ for any proper algebraic subgroup J of H, since then S/J would be $S^{\max}(A)$. If $g \in \Gamma_S(B)$ then, by the properties of $S^{\max}(B)$, up to isogeny there is $S_B \xrightarrow{p} S$ such that $g = (Tp)(b) \cdot \beta$, for some $\beta \in S(A)$. But then $g = (Tp)(e) \cdot (Tp)(\alpha) \cdot \beta$, and $(Tp)(\alpha) \cdot \beta \in S(A)$. Hence e is a basis for $\Gamma(B/A)$, and $H = S^{\max}(B/A)$. □

Definition 2.7. For an extension $A \hookrightarrow B$ in $\mathcal{K}$, with B finitely generated, define the *group rank* and *predimension* to be

$$\mathrm{grk}(B/A) = \dim S^{\max}(B/A) \qquad\qquad \delta(B/A) = \mathrm{td}(B/A) - \mathrm{grk}(B/A)$$

respectively. For any subset $X \subseteq B$, define $\mathrm{grk}(X/A) = \mathrm{grk}(\langle X, A\rangle/A)$ and define $\delta(X/A) = \delta(\langle X, A\rangle/A)$. Also define $\mathrm{grk}(A) = \mathrm{grk}(A/C)$ and $\delta(A) = \delta(A/C)$.

The Schanuel property says precisely that $\delta(A) \geqslant 0$ for each finitely generated structure A, with equality iff $A = C$.

Lemma 2.8. *For an extension $A \hookrightarrow B$ in $\mathcal{K}$, with B finitely generated,* $\mathrm{grk}(B) = \mathrm{grk}(B/A) + \mathrm{grk}(A)$ *and* $\delta(B) = \delta(B/A) + \delta(A)$.

Proof. The statement for group rank is immediate from proposition 2.6. The same property holds for transcendence degree, that is, $\mathrm{td}(B/C) = \mathrm{td}(B/A) + \mathrm{td}(A/C)$, and the result for the predimension follows. □

An essential property of δ is that it is *submodular.*

Lemma 2.9. *The predimension δ is submodular on $\mathcal{K}$. That is, for any finitely generated $B \in \mathcal{K}$ and any $A_1, A_2 \subseteq B$, such that $A_1, A_2 \in \mathcal{K}$,*

$$\delta(A_1 \cup A_2) + \delta(A_1 \cap A_2) \leqslant \delta(A_1) + \delta(A_2).$$

Proof. Let $A_0 = A_1 \cap A_2$, and $A_3 = \langle A_1 \cup A_2\rangle$. We first show that $\mathrm{grk}(A_3/A_0) \geqslant \mathrm{grk}(A_1/A_0) + \mathrm{grk}(A_2/A_0)$. For $i = 1, 2, 3$, let $S_i = S^{\max}(A_i/A_0)$ and let $g_i \in TS_i$ be a basis for $\Gamma(A_i/A_0)$. By lemma 2.4, there are (up to isogeny) homomorphisms $S_3 \xrightarrow{q_i} S_i$ for $i = 1, 2$ such that $g_i = (Tq_i)(g_3)$.

Suppose $\mathrm{grk}(A_3/A_0) < \mathrm{grk}(A_1/A_0) + \mathrm{grk}(A_2/A_0)$. Then $\dim S_3 < \dim S_1 \times S_2$, and by definition of $S^{\max}$, there is a proper algebraic subgroup H of $S_1 \times S_2$ such that (g_1, g_2) lies in an A_0-coset of TH. Now H is normal in $S_1 \times S_2$ since the groups are commutative, so it is the kernel of some algebraic group

homomorphism $S_1 \times S_2 \xrightarrow{p} J$, and $(Tp)(g_1, g_2) = \alpha \in TJ(A_0)$. Since the product $S_1 \times S_2$ is also the direct sum $S_1 \oplus S_2$ in the category of commutative algebraic groups, we can write $(Tp)(g_1, g_2)$ as $(Tp_1)(g_1) \cdot (Tp_2)(g_2)$, where $S_1 \xrightarrow{p_1} J$ is given by $p_1(x) = p(x, 1)$, and symmetrically p_2. Then $(Tp_1)(g_1) = \alpha \cdot (Tp_2)(g_2)^{-1} \in TJ(A_2)$, so $Tp_1(g_1) \in TJ(A_1 \cap A_2) = TJ(A_0)$. Thus g_1 lies in an A_0-coset of $T(\ker p_1)$, but $\dim J > 0$, so $\ker p_1$ is a proper algebraic subgroup of S_1, which contradicts g_1 being a basis of $\Gamma(A_1/A_0)$. So $\operatorname{grk}(A_3/A_0) \geqslant \operatorname{grk}(A_1/A_0) + \operatorname{grk}(A_2/A_0)$. Thus

$$\operatorname{grk}(A_1 \cup A_2/A_0) + 2\operatorname{grk}(A_0) \geqslant \operatorname{grk}(A_1/A_0) + \operatorname{grk}(A_2/A_0) + 2\operatorname{grk}(A_0)$$

and hence, by lemma 2.8,

$$\operatorname{grk}(A_1 \cup A_2) + \operatorname{grk}(A_1 \cap A_2) \geqslant \operatorname{grk}(A_1) + \operatorname{grk}(A_2). \tag{1}$$

Now

$$\operatorname{td}(A_1 \cup A_2/C) + \operatorname{td}(A_1 \cap A_2/C) \leqslant \operatorname{td}(A_1/C) + \operatorname{td}(A_2/C) \tag{2}$$

and so, subtracting (1) from (2), we see that δ is submodular. $\square$

2.4 Self-sufficient embeddings

The intuition behind the predimension function δ is that is measures the number of "degrees of freedom", which could be thought of as the number of variables minus the number of constraints. We cannot amalgamate over all embeddings because an amalgam of arbitrary embeddings will not always have the Schanuel property. That is, $\mathcal{K}$ does not have the amalgamation property. The problem is that for some embeddings $A \hookrightarrow B$ there will be extra constraints on A which are not apparent in A but are witnessed only in the extension B. We will amalgamate only over those embeddings where this does not occur. Informally, an embedding $A \hookrightarrow B$ is *self-sufficient* if any dependency (constraint) between members of A in B is already witnessed in A. The formal definition does not require the structures to be finitely generated.

Definition 2.10. We say that an embedding of structures $A \hookrightarrow B$ is *self-sufficient* iff for every $X \subseteq_{f.g.} B$ we have $\delta(X \cap A) \leqslant \delta(X)$. In this case, we write the embedding as $A \lhd B$ or $A \hookrightarrow\!\!\!\lhd\!\!\!\rightarrow B$ and we say that A is self-sufficient in B.

Lemma 2.11. *Taking all the objects of $\mathcal{K}$ with just the self-sufficient embeddings gives a subcategory $\mathcal{K}^{\lhd}$ of $\mathcal{K}$.*

Proof. It is immediate that identity embeddings are self-sufficient and the composite of self-sufficient embeddings is self-sufficient. $\square$

It is customary to write self-sufficient embeddings as $A \leqslant B$, but this seems to me to be an unnecessary duplication of a common symbol and potentially confusing, so I prefer to avoid it. This is a simplification of the original definition of a self-sufficient embedding (see for example [Hru93]), and it is equivalent to the original definition for any δ which is submodular, as predimension functions for Hrushovski-type constructions are.

Lemma 2.12. *If $A_i \lhd B$ for each i in some index set I and $A = \bigcap_{i \in I} A_i$ is the intersection in $\mathcal{K}$, then $A \lhd B$. In particular, the category $\mathcal{K}^{\lhd}$ has intersections.*

Proof. First we show that it holds for binary intersections. Suppose $A_1, A_2 \lhd B$. Let $X \subseteq_{f.g.} A_1$. Then $\delta(X \cap (A_1 \cap A_2)) = \delta(X \cap A_2) \leqslant \delta(X)$ since $A_2 \lhd B$ and $X \subseteq_{f.g.} B$. So $A_1 \cap A_2 \lhd A_1$, but also $A_1 \lhd B$ and so $A_1 \cap A_2 \lhd B$. By induction, any finite intersection of self-sufficient substructures of B is also self-sufficient in B.

The case of an arbitrary intersection of self-sufficient subsets follows by a finite character argument. Let $X \subseteq_{f.g.} B$. Then $X \cap \bigcap_{i \in I} A_i$ is an algebraically closed subfield of X, which has finite transcendence degree. The lattice of algebraically closed subfields of X has no infinite chains, hence there is a finite subset I_0 of I such that $X \cap \bigcap_{i \in I} A_i = X \cap \bigcap_{i \in I_0} A_i$. By the above, $\bigcap_{i \in I_0} A_i \lhd B$, and so $\delta(X \cap \bigcap_{i \in I} A_i) \leqslant \delta(X)$. So $\bigcap_{i \in I} A_i \lhd B$ as required. $\square$

As with $\mathcal{K}$, the existence of intersections allows one to define the subobject *generated* by some set, and consequently the notion of a finitely generated object in $\mathcal{K}^{\lhd}$. This greatly simplifies the presentation, and is one reason for working in the category $\mathcal{K}$ rather than the category of all $\mathcal{L}_{\mathcal{S}}$-substructures of models. To distinguish this notion of generation from that in $\mathcal{K}$, we give it a different name.

Definition 2.13. If B is a structure and X is a subset of B then the *hull* of X in B is given by $\lceil X \rceil = \bigcap \{A \lhd B \mid X \subseteq A\}$.

Note that as for $\langle X \rangle$, the hull $\lceil X \rceil$ depends on B, although we do not write the dependence explicitly. Hulls give another way of showing that an embedding is self-sufficient.

Lemma 2.14. *$A \lhd B$ iff for every $Y \subseteq_{f.g.} A$, $\lceil Y \rceil \subseteq A$.*

Proof. Suppose $Y \subseteq_{f.g.} A$ and $\lceil Y \rceil \not\subseteq A$. Let $X = \lceil Y \rceil$. Then $\delta(X) < \delta(X \cap A)$, so $A \not\lhd B$. Conversely, suppose $A \not\lhd B$, so there is $X \subseteq_{f.g.} B$ such that $\delta(X) < \delta(X \cap A)$. Then $X \cap A$ is finitely generated so take $Y = X \cap A$. $\square$

2.5 The amalgamation property

Lemma 2.15. *A structure is finitely generated in the sense of $\mathcal{K}^{\lhd}$ iff it is finitely generated in the sense of $\mathcal{K}$.*

Proof. The right to left direction is immediate, since for any set X, $\langle X \rangle \subseteq \lceil X \rceil$.

We show that if $B \in \mathcal{K}$ and $X \subseteq B$ is a finite subset then $\lceil X \rceil$ is finitely generated in $\mathcal{K}$. Consider $\{\delta(A) \mid X \subseteq A \subseteq_{f.g.} B\}$, a nonempty subset of $\mathbb{N}$. Let A be such that $\delta(A)$ is least. Then for any $Y \subseteq_{f.g.} B$,

$$0 \leqslant \delta(A \cup Y) - \delta(A) \leqslant \delta(Y) - \delta(A \cap Y)$$

with the first comparison holding by the minimality of $\delta(A)$ and the second by submodularity of δ. Thus $A \lhd B$. In particular, $\lceil X \rceil \subseteq A$, and so $\lceil X \rceil$ is finitely generated in $\mathcal{K}$. $\square$

We define the category $\mathcal{K}^{\lhd}_{<\aleph_0}$ to be the subcategory of $\mathcal{K}^{\lhd}$ consisting of the finitely generated structures, together with all self-sufficient embeddings. In order to apply the amalgamation theorem, we need to show that $\mathcal{K}^{\lhd}_{<\aleph_0}$ has the amalgamation property. In fact, we show more than this, which is necessary when it comes to axiomatizing the amalgam.

Proposition 2.16 (Free asymmetric amalgamation)**.** *If we have embeddings $A \lhd B_1$ and $A \hookrightarrow B_2$ in $\mathcal{K}$ then there is $E \in \mathcal{K}$ (the* free amalgam *of B_1 and*

B_2 over A) and embeddings $B_1 \hookrightarrow E$ and $B_2 \lhd E$ such that the square

$$\begin{tikzcd} & E & \\ B_1 \arrow[ur, hook] & & B_2 \arrow[ul, hook, "\lhd"'] \\ & A \arrow[ul, hook, "\lhd"] \arrow[ur, hook] & \end{tikzcd}$$

commutes, and $E = \langle B_1, B_2 \rangle$. Furthermore, if $A \lhd B_2$ then $B_1 \lhd E$.

Proof. Let β_1, β_2 be transcendence bases of B_1, B_2 over A. As a field, take E to be the algebraic closure of the extension of A with transcendence base the disjoint union $\beta_1 \sqcup \beta_2$. This defines the field E and the embeddings $B_1 \hookrightarrow E$ and $B_2 \hookrightarrow E$ uniquely up to isomorphism, because B_1 and B_2 are algebraically disjoint over A in E. For each $S \in \mathcal{S}$, define $\Gamma_S(E)$ to be the subgroup of $TS(E)$ generated by $\Gamma_S(B_1) \cup \Gamma_S(B_2)$. Axioms U1—U7 then hold by the construction.

Let X be a finitely generated algebraically closed substructure of E. Note that δ and grk were originally defined only for structures satisfying the Schanuel property, and we do not yet know that it holds for E. However, the definitions of δ and grk make sense for X because the conclusion of lemma 2.4 holds, and so $\operatorname{grk}(X)$ is well-defined and finite.

Let $S = S^{\max}(X/X \cap B_2)$, and let $g \in \Gamma_S(X)$ be a basis for $\Gamma(X/X \cap B_2)$. Then by the construction of $\Gamma_S(E)$, there are $h \in \Gamma_S(B_1)$ and $b \in \Gamma_S(B_2)$ such that $g = h \cdot b$. The group operation of S is defined over C, so certainly over B_2, and so

$$\operatorname{td}(g/X \cap B_2) \geqslant \operatorname{td}(g/B_2) = \operatorname{td}(h/B_2) = \operatorname{td}(h/A) \geqslant \operatorname{grk}(h/A)$$

with the second equation because B_1 is algebraically independent of B_2 over A and the final comparison because $A \lhd B_1$.

We now show that $\operatorname{grk}(h/A) = \dim S$. If not, then there is $a \in TS(A)$ and a proper algebraic subgroup H of S such that $h \cdot a^{-1} \in TH(B_1)$. Now $h \cdot a^{-1} = g \cdot (a \cdot b)^{-1}$, and $a \cdot b \in TS(B_2)$, so g lies in a B_2-coset of TH. This contradicts the fact that g is a basis for $\Gamma(X/X \cap B_2)$. So $\operatorname{grk}(h/A) = \dim S$, and thus

$$\delta(X/X \cap B_2) \geqslant \operatorname{td}(g/X \cap B_2) - \dim S \geqslant 0.$$

Thus $B_2 \lhd E$. The symmetric argument shows that if $A \lhd B_2$ then $B_1 \lhd E$.

Now $\delta(X \cap B_2) \geqslant 0$ because B_2 satisfies SP, so

$$\delta(X) = \delta(X/X \cap B_2) + \delta(X \cap B_2) \geqslant 0.$$

Suppose $\delta(X) = 0$. Let $e \in \Gamma_{S'}$ be a basis for $\Gamma(X \cap B_2/C)$. Then

$$0 = \delta(X) = \operatorname{td}(g/C(e)) + \operatorname{td}(e/C) - \dim S - \dim S'$$

but, since B_2 satisfies SP, either $\operatorname{td}(e/C) > \dim S'$ or $X \cap B_2 \subseteq C$. By the calculation above, $\operatorname{td}(g/C(e)) \geqslant \operatorname{td}(g/X \cap B_2) \geqslant \dim S$, so $\operatorname{td}(e/C) \leqslant \dim S'$, and hence $X \cap B_2 \subseteq C$. Thus g is independent from B_2 over C, so $g \in B_1$. But then $\operatorname{td}(g/C) = \dim S$, so $X \subseteq C$ using SP for B_1. Hence E has SP. $\square$

2.6 The amalgamation theorem

The category $\mathcal{K}^{\lhd}$ is not the category of all finitely generated models of a universal first order theory, because its objects are all algebraically closed field extensions of a fixed C. Thus we must use a more abstract version of the Fraissé amalgamation theorem than that given, for example, in [Hod93]. We use a variant of the category-theoretic version given in [DG92]. We must explain how some standard notions are translated into this setting.

Fix an ordinal λ, and consider a category $\mathcal{C}$. A *chain* of length λ in $\mathcal{C}$ is a collection $(Z_i)_{i<\lambda}$ of objects of $\mathcal{C}$ together with arrows $Z_i \xrightarrow{\gamma_{ij}} Z_j$ for each $i \leqslant j < \lambda$, such that for each i, $\lambda_{ii} = 1_{Z_i}$, and if $i \leqslant j \leqslant k < \lambda$ then $\gamma_{jk} \circ \gamma_{ij} = \gamma_{ik}$. The *union* or *direct limit* of a λ-chain is an object $Z = Z_\lambda$ with arrows $Z_i \xrightarrow{\gamma_{i\lambda}} Z$ for each $i < \lambda$, satisfying the usual universal property of a direct limit.

For λ an infinite regular cardinal, identified with its initial ordinal, an object A of $\mathcal{C}$ is said to be λ-*small* iff for every λ-chain (Z_i, γ_{ij}) in $\mathcal{C}$ with direct limit Z, any arrow $A \xrightarrow{f} Z$ factors through the chain, that is, there is $i < \lambda$ and $A \xrightarrow{f^*} Z_i$ such that $f = \gamma_{i\lambda} \circ f^*$. For example, in the category of sets a set is $\aleph_0$-small iff it is finite. Write $\mathcal{C}_{<\lambda}$ for the full subcategory of $\mathcal{C}$ consisting of all the λ-small objects of $\mathcal{C}$, and $\mathcal{C}_{\leqslant\lambda}$ for the full subcategory of $\mathcal{C}$ consisting of all unions of λ-chains of λ-small objects.

Definition 2.17. We say that $\mathcal{C}$ is a λ-*amalgamation category* iff the following hold.

- Every arrow in $\mathcal{C}$ is a monomorphism.
- $\mathcal{C}$ has direct limits (unions) of chains of every ordinal length up to λ.
- $\mathcal{C}_{<\lambda}$ has at most λ objects up to isomorphism.
- For each object $A \in \mathcal{C}_{<\lambda}$ there are at most λ extensions of A in $\mathcal{C}_{<\lambda}$, up to isomorphism.
- $\mathcal{C}_{<\lambda}$ has the *amalgamation property* (AP), that is, any diagram of the form

$$\begin{tikzcd} B_1 & & B_2 \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

  can be completed to a commuting square

$$\begin{tikzcd} & C & \\ B_1 \arrow[ur] & & B_2 \arrow[ul] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

  in $\mathcal{C}_{<\lambda}$.

- $\mathcal{C}_{<\lambda}$ has the *joint embedding property* (JEP), that is, for every $B_1, B_2 \in \mathcal{C}_{<\lambda}$ there is $C \in \mathcal{C}_{<\lambda}$ and arrows

$$\begin{tikzcd} & C & \\ B_1 \arrow[ur] & & B_2 \arrow[ul] \end{tikzcd}$$

  in $\mathcal{C}_{<\lambda}$.

An *extension* of A is simply an arrow with domain A. To say that two extensions $A \xrightarrow{f} B$ and $A \xrightarrow{f'} B'$ are isomorphic means that there is an isomorphism $B \xrightarrow{g} B'$ such that $f' = gf$. In [DG92], Droste and Göbel

consider a stronger condition than bounding the number of extensions of each A, namely that for any pair of objects A and B there are only λ arrows from A to B. This allows them to use the pre-existing definition of a λ-algebroidal category, but it is not strong enough for our purposes. For example, if A is a pure algebraically closed field extension of C of transcendence degree one then there are $2^{\aleph_0}$ embeddings of A into itself over C, but they are all isomorphisms, and hence isomorphic extensions. The condition bounding only the number of extensions is model-theoretically much more natural.

To say that an object U of $\mathcal{C}$ is $\mathcal{C}_{\leqslant\lambda}$-universal means that for every object $A \in \mathcal{C}_{\leqslant\lambda}$ there is an arrow $A \longrightarrow U$ in $\mathcal{C}$. To say that U is $\mathcal{C}_{<\lambda}$-saturated means that for any $A, B \in \mathcal{C}_{<\lambda}$ and any arrows $A \xrightarrow{f} U$ and $A \xrightarrow{g} B$ there is an arrow $B \xrightarrow{h} U$ such that $h \circ g = f$. These are just the translations into category-theoretic language of the usual model-theoretic notions.

Theorem 2.18 (Amalgamation theorem)**.** *If $\mathcal{C}$ is a λ-amalgamation category then there is an object $U \in \mathcal{C}_{\leqslant\lambda}$, the "Fraissé limit", which is $\mathcal{C}_{\leqslant\lambda}$-universal and $\mathcal{C}_{<\lambda}$-saturated. Furthermore, U is unique up to isomorphism.*

Proof. The proof in [DG92] goes through, even with the slightly weaker hypothesis bounding the number of extensions rather than the number of arrows. □

The notion of $\aleph_0$-small is the same as finitely generated in our example.

Lemma 2.19. *An object A of $\mathcal{K}$ is $\aleph_0$-small in $\mathcal{K}$ or in $\mathcal{K}^{\triangleleft}$ iff it is finitely generated (that is, iff* $\mathrm{td}(A/C)$ *is finite).*

Proof. If A is finitely generated by $x_1, \ldots, x_n$ and $A \hookrightarrow Z$ where Z is the union of an ω-chain $(Z_i)_{i<\omega}$ then each x_j lies in some $Z_{i(j)}$, so taking i greater than each $i(j)$ the embedding factors through Z_i. This argument works for both categories $\mathcal{K}$ and $\mathcal{K}^{\triangleleft}$.

Conversely, if $\mathrm{td}(A/C)$ is infinite, let $X \cup \{x_j\}_{j<\omega}$ be an transcendence base for A over C, and let $Z_i = \langle X \cup \{x_j \mid j \leqslant i\}\rangle$. Then A is the union of the chain (Z_i) in $\mathcal{K}$, but is not equal to any of the Z_i. Hence it is not $\aleph_0$-small in $\mathcal{K}$.

Now let $W_i = \lceil X \cup \{x_j \mid j \leqslant i\}\rceil$. By lemma 2.15 together with the existence of free amalgams, $\mathrm{td}(\lceil B\rceil/B)$ is finite for any B. Thus W_i is an ω-chain in $\mathcal{K}^{\triangleleft}$, with a strictly increasing cofinal subchain and union A, and so A is not $\aleph_0$-small in $\mathcal{K}^{\triangleleft}$. □

Lemma 2.20. *Let $A \in \mathcal{K}$ and let B be a self-sufficient extension of A which is finitely generated over A. Then B is determined up to isomorphism by $S^{\max}(B/A)$, the algebraic locus $\mathrm{Loc}_A(g)$ of a basis g for $\Gamma(B/A)$, and the natural number $\mathrm{td}(B/A(g))$.*

Proof. As a field extension, B is determined by its transcendence degree over A. By lemma 2.4, the points of $\Gamma_S(B)$ for each $S \in \mathcal{S}$ are determined by $S^{\max}(B/A)$ and the basis g. $\square$

Proposition 2.21. *$\mathcal{K}^{\lhd}$ is an $\aleph_0$-amalgamation category.*

Proof. Every embedding in $\mathcal{K}^{\lhd}$ is certainly a monomorphism, because $\mathcal{K}^{\lhd}$ is a concrete category and the underlying function is injective. It is also easy to see that $\mathcal{K}^{\lhd}$ has unions of chains of any ordinal length, and in particular unions of ω-chains.

There are only countably many $S \in \mathcal{S}$, and only countably many algebraic varieties defined over A, so by lemma 2.20 there are only countably many self-sufficient extensions of A. The structure C embeds self-sufficiently into every $B \in \mathcal{K}^{\lhd}$, so taking $A = C$ it follows in particular that $\mathcal{K}^{\lhd}_{<\aleph_0}$ has only countably many objects. The amalgamation property for $\mathcal{K}^{\lhd}_{<\aleph_0}$ is given by proposition 2.16, and the joint embedding property follows from the amalgamation property, again since C embeds self-sufficiently into each $B \in \mathcal{K}$. $\square$

Putting proposition 2.21 and theorem 2.18 together, we get the universal structure we want.

Theorem 2.22. *There is a countable model U of $T^U_{\mathcal{S}}$ which is universal and saturated with respect to self-sufficient embeddings. Furthermore, U is unique up to isomorphism.* $\square$

Note that this Fraissé limit U is a union of an ω-chain of countable structures, hence is countable. Every countable model of $T^U_{\mathcal{S}}$ can be self-sufficiently embedded in some $A \in \mathcal{K}_{\leqslant\aleph_0}$, by extending the constant field and taking the algebraic closure. Thus the $\mathcal{K}^{\lhd}_{\leqslant\aleph_0}$-universality of U implies that every countable model of $T^U_{\mathcal{S}}$ can be self-sufficiently embedded into U. Similarly, U is saturated with respect to self-sufficient embeddings for any self-sufficient substructures of finite transcendence degree.

2.7 Pregeometry and dimension

The geometry of the Fraissé limit U is controlled by a pregeometry, which we now describe. For any model M of $T^U_{\mathcal{S}}$, in particular U, the predimension function δ gives rise to a *dimension* notion on M. The dimension function is conventially denoted d and is defined as follows.

Definition 2.23. For $X \subseteq_{\text{fin}} M$ (or even $X \subseteq M$ with $\operatorname{td}(X/C)$ finite), define $d(X) = \delta(\lceil X \rceil)$ or, equivalently, $d(X) = \min\{\delta(XY) \mid Y \subseteq_{\text{fin}} M\}$. For X as above and any $A \subseteq M$, the dimension of X over A is defined to be

$$d(X/A) = \min\{d(XY) - d(Y) \mid Y \subseteq_{\text{fin}} A\}.$$

Note that $d(X) = d(X/\emptyset)$, so the two definitions agree.

Lemma 2.24 (Properties of d)**.** *Let $X, Y \subseteq_{\text{fin}} M$ and $A, B \subseteq M$.*

1. *If $X \subseteq Y$ then $d(X/A) \leqslant d(Y/A)$.*
2. *If $A \subseteq B$ then $d(X/A) \geqslant d(X/B)$.*
3. *d is submodular: $d(XY) + d(X \cap Y) \leqslant d(X) + d(Y)$.*
4. *$d(X/Y) = d(XY) - d(Y)$.*
5. *$d(X) \geqslant 0$, with equality iff $X \subseteq C$.*
6. *For any $x \in M$, $d(x/A) = 0$ or 1.*

Proof. The first two parts are immediate from the definition. For submodularity:

$$\begin{aligned} d(XY) + d(X \cap Y) &= \delta(\lceil XY \rceil) + \delta(\lceil X \cap Y \rceil) \\ &\leqslant \delta(\lceil X \rceil \lceil Y \rceil) + \delta(\lceil X \rceil \cap \lceil Y \rceil) \\ &\leqslant \delta(\lceil X \rceil) + \delta(\lceil Y \rceil) \\ &\quad = d(X) + d(Y) \end{aligned}$$

For part 4, let $Z \subseteq Y$. Then

$$d(XY) - d(Y) \leqslant d(XZ) - d(XZ \cap Y) \leqslant d(XZ) - d(Z)$$

by submodularity and monotonicity of d. Thus the minimum value of $d(XZ) - d(Z)$ occurs when $Z = Y$.

Part 5 follows from the Schanuel property.

For part 6, take $A_0 \subseteq_{\text{fin}} A$ such that $d(x/A) = d(x/A_0)$. Then

$$\begin{array}{rcl} d(x/A_0) & = & d(A_0x) - d(A_0) \\ & = & \delta(\lceil A_0x \rceil) - \delta(\lceil A_0 \rceil) \\ & = & \delta(\lceil \lceil A_0 \rceil x \rceil) - \delta(\lceil A_0 \rceil) \\ & \leqslant & \delta(\lceil A_0 \rceil x) - \delta(\lceil A_0 \rceil) \\ & \leqslant & \operatorname{td}(x/\lceil A_0 \rceil) \leqslant 1 \end{array}$$

so $d(x/A_0) = 0$ or 1. □

Proposition 2.25. *The operator* $\mathcal{P}M \xrightarrow{\text{cl}} \mathcal{P}M$ *given by* $x \in \operatorname{cl} A \iff d(x/A) = 0$ *is a pregeometry on* M*. If* $X \subseteq M$ *is such that* $d(X)$ *is defined (that is,* $\operatorname{td}(X/C)$ *is finite) then* $d(X)$ *is equal to the dimension of* X *in the sense of the pregeometry.*

Proof. It is straightforward to check that cl is a closure operator with finite character. It remains to check the exchange property. Let $A \subseteq M, a, b \in M$, and $a \in \operatorname{cl}(Ab) \smallsetminus \operatorname{cl}(A)$. By finite character, there is a finite $A_0 \subseteq A$ such that $a \in \operatorname{cl}(A_0b)$. Then $d(a/A_0) = 1$. Using part 4 of lemma 2.24, we have

$$\begin{array}{rcl} d(b/A_0a) & = & d(A_0ab) - d(A_0a) \\ & = & d(A_0b) - d(A_0a) \\ & = & [d(A_0) + d(b/A_0)] - [d(A_0) + d(a/A_0)] \\ & = & [d(A_0) + 1] - [d(A_0) + 1] = 0 \end{array}$$

and so $b \in \operatorname{cl}(Aa)$.

Finally, x is independent from A iff $d(x/A) = 1$, and so d agrees with the dimension coming from the pregeometry. □

From now on, by the *dimension* of a structure $A \in \mathcal{K}$ we mean the dimension in the sense of this pregeometry on A. Note that self-sufficient embeddings are precisely those embeddings which preserve the dimension.

2.8 Freeness and Rotundity

To explain what the theory of the structure U is, we must translate $\mathcal{K}^{\lhd}_{<\aleph_0}$-saturation into a more tractable form. We will show that it is equivalent to saying that certain algebraic subvarieties of TS have a nonempty intersection with Γ_S.

Definition 2.26. An irreducible subvariety V of TS is *free* iff V is not contained in a coset of TH for any proper algebraic subgroup H of S. It is *absolutely free* iff $\mathrm{pr}_S V$ is not contained in a coset of any such H and $\mathrm{pr}_{LS} V$ is not contained in a coset of LH for any such H.

A point $g \in TS$ is (*absolutely*) *free* over a field A iff $\mathrm{Loc}_A(g)$ is (absolutely) free.

An irreducible subvariety V of TS is *rotund* iff for every quotient map $S \xrightarrow{f} H$,

$$\dim(Tf)(V) \geqslant \dim H$$

and *strongly rotund* iff for every such f with $H \neq 1$,

$$\dim(Tf)(W) \geqslant \dim H + 1.$$

A point $g \in TS$ is (*strongly*) *rotund* over a field A iff $\mathrm{Loc}_A(g)$ is (strongly) rotund. A reducible variety is (*strongly*) *rotund* iff at least one of its irreducible components is.

Lemma 2.27. *Let $A \lhd B$ be a self-sufficient extension in $\mathcal{K}^{\lhd}$, with B finitely generated over A. Let $S = S^{\max}(B/A)$ and let $g \in \Gamma_S$ be a basis for B over A.*

Then g is free over A, absolutely free over C, rotund over A, and strongly rotund over C.

Proof. If $S = 1$ then the result is trivial. Assume $S \neq 1$. By the definition of a basis, g does not lie in an A-coset of TH for any proper algebraic subgroup H of S, and hence $\mathrm{Loc}_A(g)$ is not contained in such a coset. By axioms U4 and U5, $\mathrm{pr}_{LS} \mathrm{Loc}_C(g)$ lies in a coset of LH iff $\mathrm{pr}_S \mathrm{Loc}_C(g)$ lies in a coset of H, since $g \in \Gamma_S$. If both held then g would lie in a C-coset of TH, but it does not, so g is absolutely free over C.

For each quotient map $S \xrightarrow{f} H$,

$$\dim((Tf)(\mathrm{Loc}_A g)) - \dim H = \dim(\mathrm{Loc}_A((Tf)(g)) - \dim H = \delta((Tf)(g)/A) \geqslant 0$$

as g is free over A and $A \lhd B$, so g is rotund over A.

Similarly, if $H \neq 1$ then

$$\dim((Tf)(\mathrm{Loc}_C g)) - \dim H = \dim(\mathrm{Loc}_C((Tf)(g)) - \dim H = \delta((Tf)(g)/C) \geqslant 1$$

as B satisfies the Schanuel property, so g is strongly rotund over C. □

It is useful to isolate the subvarieties which occur as the locus of a basis of $\Gamma(B/A)$ for an extension B/A which cannot be split into a tower of smaller extensions. We call these *perfectly rotund* subvarieties.

Definition 2.28. A subvariety V of TS is *perfectly rotund* iff it is rotund, $\dim V = \dim S$, and for every proper, nontrivial quotient map $S \xrightarrow{f} H$,

$$\dim(Tf)(V) > \dim H.$$

2.9 Existential closedness

Definition 2.29. Let X be a variety. Any constructible set P and Zariski-closed $V \subseteq X \times P$ defines a *parametric family* $(V_p)_{p\in P}$ of subvarieties of X, where V_p is the fibre above p of the natural projection $X \times P \to P$, restricted to V. We write $(V_p)_{p\in P(C)}$ to be the fibres over the C-points of P, and also call this a parametric family.

Definition 2.30. We consider three forms of Existential Closedness, and two notions relating to dimension: Non-Triviality and Infinite Dimensionality, for a model M of $T^U_{\mathcal{S}}$.

EC For each $S \in \mathcal{S}$, each irreducible rotund subvariety V of TS, and each parametric family $(W_e)_{e\in Q(C)}$ of proper subvarieties of V, with Q a constructible set defined over C_0, there is $g \in \Gamma_S \cap V \smallsetminus \bigcup_{e\in Q(C)} W_e$.

EC′ The same as EC except only for perfectly rotund V.

SEC (Strong existential closedness) For each $S \in \mathcal{S}$, each rotund subvariety V of TS, and each finitely generated field of definition A of V, the intersection $\Gamma_S \cap V$ contains a point which is generic in V over $A \cup C$.

NT There is $x \in M$ such that $x \notin C$.

ID The structure M is infinite dimensional.

NT is equivalent to saying that the dimension of M is nonzero, so it is implied by ID. Clearly SEC implies EC and EC implies EC′. We prove that EC′ implies EC using of the tool of intersecting a variety with generic

hyperplanes. For $p = (p_1, \dots, p_N) \in \mathbb{A}^N \smallsetminus \{0\}$, let the hyperplane Π_p in the affine space $\mathbb{A}^N$ be given by

$$x \in \Pi_p \quad \text{iff} \quad \sum_{i=1}^{N} p_i x_i = 1.$$

Consider the family of hyperplanes $(\Pi_p)_{p \in \mathbb{A}^N \smallsetminus \{0\}}$, which is the family of all affine hyperplanes which do not pass through the origin. From the equation defining the hyperplanes it follows that there is a duality: $a \in \Pi_p$ iff $p \in \Pi_a$.

The lemma we use is in the style of model-theoretic geometry, and is adapted from part of a proof in [Zil04]. Here and later, $x \in \operatorname{acl} X$ means that x is a point *in some variety* which is algebraic over X. We do not restrict this notation to x being an element or tuple from affine space.

Lemma 2.31. *Let A be a field, let $g \in \mathbb{A}^N$ and let p be generic in Π_g over A. Suppose that h is any tuple (a point in any algebraic variety) such that $h \in \operatorname{acl}(Ag)$. Then either $g \in \operatorname{acl}(Ah)$ or $\operatorname{td}(h/Ap) = \operatorname{td}(h/A)$ (that is, h is independent of p over A).*

Proof. If g is algebraic over A then the result is trivial, so we assume not.

Let $U = \operatorname{Loc}(p/\operatorname{acl}(Ah))$. Suppose $\operatorname{td}(h/Ap) < \operatorname{td}(h/A)$. Then, by counting transcendence bases, $\dim U = \operatorname{td}(p/Ah) < \operatorname{td}(p/A) = N$, the last equation holding because $g \notin \operatorname{acl}(A)$ and so p is generic in $\mathbb{A}^N$ over A. But $\operatorname{td}(p/Ah) \geqslant \operatorname{td}(p/Ag) = N - 1$ as p is generic in Π_g, an $(N-1)$-dimensional variety defined over Ag. Hence $\dim U = N - 1$. Now $\operatorname{acl}(Ah) \subseteq \operatorname{acl}(Ag)$, so $U = \operatorname{Loc}(p/\operatorname{acl}(Ah) \supseteq \operatorname{Loc}(p/\operatorname{acl}(Ag)) = \Pi_g$. But $\dim U = \dim \Pi_g$ and both U and Π_g are irreducible and Zariski-closed in $\mathbb{A}^N$, so $U = \Pi_g$.

Hence Π_g is defined over $\operatorname{acl}(Ah)$, and so is the set

$$\left\{ x \in \mathbb{A}^N \mid (\forall y \in \Pi_g)[x \in \Pi_y] \right\} = \{g\}.$$

Thus $g \in \operatorname{acl}(Ah)$. $\square$

To have generic hyperplanes definable in the structure, we need to know that it has large enough transcendence degree.

Lemma 2.32. *Suppose $\mathcal{S} \neq \{1\}$ and $M \models T_{\mathcal{S}}^{U} + \mathrm{NT} + \mathrm{EC}'$. Then $\operatorname{td}(M/C)$ is infinite.*

Proof. We build a tower of algebraically closed field extensions $C \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots$ inside M. By NT, we can find a proper extension K_1 of C.

Now suppose inductively that we have built the tower up to K_i for some $i \geqslant 1$. Let $S \in \mathcal{S}$ be nontrivial, and let $x_i \in LS(K_i) \smallsetminus LS(K_{i-1})$. Let V_i be the subvariety of TS given by $V_i = \{(x, y) \in LS \times S \mid x = x_i\}$. Then V_i is perfectly rotund, so, by EC′, there is $y_i \in S(M)$ such that $(x_i, y_i) \in \Gamma_S$. Let $K_{i+1} = K_i(y_i)^{\mathrm{alg}}$. By SP, $\operatorname{td}(K_{i+1}/C) \geqslant i \dim S + 1$ for each i. Thus $\operatorname{td}(M/C)$ is infinite. □

Proposition 2.33. EC′ $\Longrightarrow$ EC.

Proof. The proof is a sequence of reductions. Suppose $M \models T^U_{\mathcal{S}} + \mathrm{EC}'$. If $M = C$ or $\mathcal{S} = \{1\}$ then trivially $M \models \mathrm{EC}$, so we assume $M \models \mathrm{NT}$ and $\mathcal{S} \neq \{1\}$.

Let $S \in \mathcal{S}$, and let $V \subseteq TS$ be rotund. We may assume that V is irreducible.

Step 1: $\dim \mathbf{V} = \dim \mathbf{S}$ We first show that if $\dim V > \dim S$, we can find a subvariety V' of V which is still rotund and irreducible, with $\dim V' = \dim V - 1$. By induction, we can assume that $\dim V = \dim S$.

Let A be a subfield of M which is a field of definition of V, with finite transcendence degree over C. Let $g \in V(M)$, generic over A. (Such a g exists because M is algebraically closed and has infinite transcendence degree over C, but we don't assume $g \in \Gamma_S$.)

Although TS will not in general be an affine variety, we can embed it in some affine space $\mathbb{A}^N$ as a constructible set in a way which preserves the notion of algebraic dependence. (This follows from the model-theoretic definition of a variety.) Now we consider g as a point in $\mathbb{A}^N$, and choose p in $\Pi_g(M)$ such that $p_1, \ldots, p_{N-1}$ are algebraically independent over $A(g)$.

Let $A' = A(p)^{\mathrm{alg}}$ and let $V' = \operatorname{Loc}(g/A')$, the locus being meant as a subvariety of V, not of $\mathbb{A}^N$. Then $\dim V' = \dim V - 1$. We show that V' is rotund.

Let $S \xrightarrow{q} H$ be an algebraic quotient map, and consider the image $h = (Tq)(g)$ in TH. Then $h \in \operatorname{acl}(Ag)$, and $\dim(Tq)(V') = \operatorname{td}(h/A')$. If $g \in \operatorname{acl}(Ah)$ then

$$\operatorname{td}(h/A') = \operatorname{td}(g/A') = \dim V - 1 \geqslant \dim S \geqslant \dim H.$$

Otherwise, by lemma 2.31, $\operatorname{td}(h/A') = \operatorname{td}(h/A)$, so $\dim(Tq)(V') = \dim(Tq)(V)$ which is at least $\dim H$ by rotundity of V. Thus V' is rotund.

Step 2: Perfect Rotundity Now we have V rotund, irreducible, and of dimension equal to $\dim S$. Again, let A be a subfield of M which is a field of definition of V, of finite transcendence degree over C, and now assume A is algebraically closed.

Consider the extension B of A where $B = A(g)^{\mathrm{alg}}$, and $g \in \Gamma_S \cap V$, with g generic in V over A. The extension $A \hookrightarrow B$ is self-sufficient, since V is rotund. Also $\delta(B/A) = 0$, as $\dim V = \dim S$. Split the extension up into a maximal chain of self-sufficient extensions

$$A = B_0 \lhd B_1 \lhd B_2 \lhd \cdots \lhd B_l = B$$

with each B_i algebraically closed and each inclusion proper. We show inductively that B_i is realised in M over B_{i-1}.

Let b_i be a basis for $\Gamma(B_i/B_{i-1})$. We have $\delta(B_i/B_{i-1}) = 0$, because $B_{i-1} \lhd B_i$ and $B_i \lhd B$, so $\mathrm{Loc}(b_i/B_{i-1})$ is free and rotund, and its dimension is equal to $\dim S^{\max}(B_i/B_{i-1})$. If $S^{\max}(B_i/B_{i-1}) \xrightarrow{q} H$ were a proper nontrivial quotient and $\dim(Tq)(\mathrm{Loc}(b_i/B_{i-1}) = \dim H$ then $B_{i-1}((Tq)(b_i))$ would be a self-sufficient extension intermediate between B_{i-1} and B_i. By assumption, no intermediate extensions exist, and so $\mathrm{Loc}(b_i/B_{i-1})$ is perfectly rotund.

Let $(W_e)_{e \in Q(C)}$ be a parametric family of proper subvarieties of V, the family defined over C_0, and $S \overset{f}{\twoheadrightarrow} S^{\max}(B_1/A)$. Then $Tf(W_e)_{e \in Q(C)}$ is a parametric family of proper subvarieties of $Tf(V)$. Since f is defined over C_0, so is this family. Hence, by EC$'$, there is $b_1' \in Tf\left(V \smallsetminus \bigcup_{e \in Q(C)} W_e\right)$. Replacing A by B_1, we inductively construct $b' \in V \smallsetminus \bigcup_{e \in Q(C)} W_e$. Thus $M \models \mathrm{EC}$. □

Proposition 2.34. *The Fraissé limit U satisfies SEC and ID.*

Proof. Let V be a rotund subvariety of TS, defined over a finitely generated subfield A of U. Let g be a generic point of V over A. Let B be the extension of A defined by taking g as a basis for $\Gamma(B/A)$. Since V is rotund, the extension $A \hookrightarrow B$ is self-sufficient.

The hull $\lceil A \rceil$ of A has finite transcendence degree over A, so by theorem 2.16 there is a free amalgam E of $\lceil A \rceil$ and B over A such that $\lceil A \rceil \lhd E$. Hence, by the $\mathcal{K}^{\lhd}_{<\aleph_0}$-saturation of U, there is an embedding θ of E into U over $\lceil A \rceil$. Then $\theta(g) \in \Gamma_S \cap V$, so U satisfies SEC.

For $n \in \mathbb{N}$, let A_n be an algebraically closed field extension of transcendence degree n over C, and for each $S \in \mathcal{S}$, let $\Gamma_S = TS(C)$. So there are

no points of Γ outside C. Then each $A_n \in \mathcal{K}$, so it embeds self-sufficiently into U. The dimension of A_n is n, and self-sufficient embeddings preserve the dimension, so the dimension of U is at least n for every $n \in \mathbb{N}$. Hence it is infinite. $\square$

Theorem 2.35. *The Fraissé limit U is the unique countable model of $T_{\mathcal{S}}^U$ which satisfies SEC and ID and has $\mathrm{td}(C/C_0) = \aleph_0$.*

Proof. The case where $\mathcal{S} = \{1\}$ is trivial, so we assume $\mathcal{S} \neq \{1\}$. Let M be any such model. We will show that M is $\mathcal{K}^{\lhd}_{<\aleph_0}$-saturated. The result follows by theorem 2.22. Let A be a self-sufficient finitely generated substructure of M and let $A \lhd B$ be a self-sufficient extension with B finitely generated. We must show that B can be embedded self-sufficiently in M over A.

By lemma 2.20, the extension B of A is determined by the group $S = S^{\max}(B/A)$, the locus $\mathrm{Loc}_A(g) \subseteq S^{\max}(B/A)$ of a basis g for $\Gamma(B/A)$, and the natural number $t = \mathrm{td}(B/A(g))$. Suppose that b is a transcendence base for $B/A(g)$. Take $S' \in \mathcal{S}$ of dimension at least t and extend b to an algebraically independent tuple $b' \in \mathbb{G}_{\mathrm{a}}{}^{\dim S'}$. Take $s \in S'$ generic over $B(b')$. Then there is a self-sufficient extension $B \hookrightarrow\!\!\lhd\!\!\rightarrow B'$ generated by (b', s) such that $(b', s) \in \Gamma_{S'}$. By replacing B by B', and S by $S \times S'$, we may assume that $t = 0$, that is, that B is generated by g over A.

Let $V = \mathrm{Loc}_A(g)$. Then V is rotund and irreducible. We use the method of step 1 of the proof of 2.33 above to reduce to replace V by a subvariety V' with $\dim V' = \dim S^{\max}(B/A)$, with V' also rotund and irreducible. However, for each generic hyperplane Π_p, by ID we may choose the $p_1, \ldots, p_{N-1}$ not just to be algebraically independent, but in fact cl-independent. Let A' be the extension of A generated by all the p_i, for each hyperplane used. Then A' is generated over A by cl-independent elements, and hence $A \lhd A'$ and $A' \lhd M$.

By SEC, there is $h \in \Gamma_S \cap V'$ in M, generic in V' over A'. Thus h is also generic in V over A. Let $B'' = \langle A'h \rangle$. Then $\delta(B''/A') = 0$, and so $B'' \lhd M$. Also $B' := \langle Ah \rangle$ is isomorphic to B over A, and $B' \lhd M$ as $\mathrm{td}(B''/B') = d(B''/B')$. Hence M is $\mathcal{K}^{\lhd}_{<\aleph_0}$-saturated, and $M \cong U$, as required. $\square$

Definition 2.36. Let $T_{\mathcal{S}}$ be the theory $T_{\mathcal{S}}^U$ + EC + NT, that is, U1 — U7 + SP + EC + NT.

We have already seen (2.1) that U1 — U7 are expressible as first order axiom schemes, and NT is a first order axiom. In section 4 we will show that SP and EC are also expressible as first order schemes, so $T_{\mathcal{S}}$ is axiomatizable as a first order theory. We will also show that $T_{\mathcal{S}}$ is complete.

3 Reducts of differential fields

3.1 Differential forms in differential algebra

Given a field C and a C-algebra A, we form the A-module $\Omega(A/C)$ of Kähler differentials as in [Sha94] or [Eis95, p386]. If A is a field, F, we can identify the F-vector space $\mathrm{Der}(F/C)$ of derivations on F which are constant on C with the dual space of $\Omega(F/C)$, by means of the universal property of Ω. If $\omega \in \Omega(F/C)$ and $D \in \mathrm{Der}(F/C)$ we write D^* for the associated element of $\Omega(F/C)^*$.

Let V be an irreducible affine variety defined over a field C, and let A be the coordinate ring of V, a C-algebra. If F is a field extension of C, an F-point x of V is associated with a C-algebra homomorphism $A \xrightarrow{x} F$, and by functoriality of Ω this defines a map

$$\begin{array}{rcl} \Omega(A/C) & \xrightarrow{x_*} & \Omega(F/C) \\ \omega & \longmapsto & \omega(x) \end{array}$$

More generally, if V is not affine (for example, V is an abelian variety) we replace A by the sheaf of coordinate rings on V, and consider the module of global differentials which we write $\Omega[V]$. Again, an F-point of V defines a map

$$\begin{array}{rcl} \Omega[V] & \xrightarrow{x_*} & \Omega(F/C) \\ \omega & \longmapsto & \omega(x) \end{array}$$

Allowing x to vary over $V(F)$ gives a map

$$\begin{array}{rcl} V(F) \times \Omega[V] & \longrightarrow & \Omega(F/C) \\ (x, \omega) & \longmapsto & \omega(x) \end{array}$$

and fixing ω gives a map which we write $V(F) \xrightarrow{\omega} \Omega(F/C)$.

If V is a commutative algebraic group G then $\Omega[G]$ is spanned by a basis of invariant differential forms. These forms are related to the logarithmic derivative.

Lemma 3.1. *If $\zeta \in \Omega[G]$ is an invariant differential form then the map*

$$\begin{array}{rcl} G(F) & \xrightarrow{\zeta} & \Omega(F/C) \\ x & \longmapsto & \zeta(x) \end{array}$$

is a group homomorphism.

If $\zeta_1, \ldots, \zeta_n$ is a basis of invariant forms of $\Omega[G]$, then the logarithmic derivative $lD_G(x) = \langle D^*\zeta_1(x), \ldots, D^*\zeta_n(x)\rangle$.

Proof. This is a restatement of the last result from §3 of [Mar00]. The first part is due to Rosenlicht [Ros57]. □

In [Ax71], Ax used the *Lie derivative* without naming or defining it explicitly, and we will use it for the same purpose. Many differential geometry books give an account of the Lie derivative in that context, but for clarity we include a description for this algebraic context.

Rewriting $\Omega(F/C)$ as $\Omega^1(F/C)$, the map $F \xrightarrow{d} \Omega^1(F/C)$ can be thought of as the coboundary map in the de Rham complex

$$0 \longrightarrow F = \Omega^0(F/C) \xrightarrow{d} \Omega^1(F/C) \xrightarrow{d} \Omega^2(F/C) \xrightarrow{d} \cdots \quad .$$

We write $\Omega^\bullet(F/C)$ for the union of the complex.

For any derivation $D \in \mathrm{Der}(F/C)$, the map $\Omega^1(F/C) \xrightarrow{D^*} F$ defined previously extends to a map $\Omega^\bullet(F/C) \xrightarrow{D^*} \Omega^\bullet(F/C)$ which is defined for $\omega \in \Omega^n(F/C)$ by

$$(D^*\omega)(D_1, \ldots, D_{n-1}) = \omega(D, D_1, \ldots, D_{n-1}).$$

This map D^* has degree -1, that is if $\omega \in \Omega^n(F/C)$ then $D^*\omega \in \Omega^{n-1}(F/C)$. By definition, d has degree $+1$. These operations can be combined into an operation of degree 0

$$L_D = D^* \circ d + d \circ D^*$$

called the *Lie derivative* of D on $\Omega^\bullet(F/C)$.

Lemma 3.2. *The Lie derivative L_D has the following properties. Let $\omega \in \Omega^1(F/C)$, $D, D' \in \mathrm{Der}(F/C)$, and $a \in F$.*

1. *L_D is C-linear.*

2. $(L_D\omega)D' = D(\omega D') - \omega[D, D']$

3. $L_D(a\omega) = (Da)\omega + a(L_D\omega)$

Proof. 1. is immediate, since d and D^* are C-linear. For 2,

$$\begin{aligned}(L_D\omega)D' &= (D^*d\omega)D' + (d(\omega D))D' \\ &= (d\omega)(D, D') + D'(\omega D) \\ &= D(\omega D') - D'(\omega D) - \omega[D, D'] + D'(\omega D) \\ &= D(\omega D') - \omega[D, D']\end{aligned}$$

and for 3,

$$\begin{aligned}L_D(a\omega)D' &= D(a\omega D') - a\omega[D, D'] \\ &= (Da)\omega D' + aD(\omega D') - a\omega[D, D'] \\ &= (Da)\omega D' + a(L_D\omega)D'.\end{aligned}$$

□

A standard fact which we need is that invariant differential forms are closed in the sense of de Rham cohomology.

Lemma 3.3. *Let G be a commutative algebraic group defined over C. Let $\omega \in \Omega[G]$ be an invariant differential form on G, and let $x \in G(F)$. Then $\omega(x)$ is a closed Kähler differential in $\Omega(F/C)$, that is, $d\omega(x) = 0$ in $\Omega^2(F/C)$.*

We give a proof for completeness. See for example [Mar00] for notation.

Proof. The Lie algebra L of $G(C)$ is canonically isomorphic to the space of invariant vector fields on $G(C)$, and is a C-vector space of dimension $n = \dim G$. Let $X_1, \ldots, X_n$ be a basis of L. The vector space $\mathrm{Der}(F/C)$ is canonically isomorphic to the space of all F-valued invariant vector fields on $G(C)$, which is $L \otimes_C F$, so $X_1, \ldots, X_n$ also forms an F-basis of $\mathrm{Der}(F/C)$. Let $D_1, D_2 \in \mathrm{Der}(F/C)$, say $D_1 = \sum_{i=1}^n a_i X_i$ and $D_2 = \sum_{i=1}^n b_i X_i$ with the $a_i, b_i \in F$. Then

$$\begin{aligned}d\omega(D_1, D_2) &= d\omega\left(\sum_{i=1}^n a_i X_i, \sum_{i=1}^n b_i X_i\right) \\ &= \sum_{i,j} a_i b_j d\omega(X_i, X_j) \quad \text{by bilinearity of } d\omega \\ &= \sum_{i,j} a_i b_j (X_i(\omega X_j) - X_j(\omega X_i) - \omega[X_i, X_j]) \quad .\end{aligned}$$

Now ω and X_j are both invariant, so for any $x, y \in G(C)$,

$$\begin{aligned}(\omega X_j)_{xy} &= \omega_{xy}(X_j)_{xy} \\ &= \omega_y(d\lambda_x^{x^{-1}}y(X_j)_{xy}) \\ &= \omega_y(d\lambda_x^{x^{-1}}yd\lambda_y^x(X_j)_y) \\ &= \omega_y(X_j)_y \\ &= (\omega X_j)_y\end{aligned}$$

and so ωX_j is a constant scalar field on $G(C)$. Thus $X_i(\omega X_j) = 0$, and similarly $X_j(\omega X_i) = 0$. So

$$d\omega(D_1, D_2) = -\sum_{i,j} a_i b_j \omega[X_i, X_j]$$

but [,] is the bracket on the Lie algebra of G, and G is commutative so the bracket is identically zero. So $d\omega(D_1, D_2) = 0$ for all $D_1, D_2 \in \mathrm{Der}(F/C)$, and hence $d\omega = 0$. $\square$

3.2 The algebraic axioms

The vector space $\Omega[G]$ is associated with the cotangent space of G at the identity, that is, with the dual of LG. Thus the canonical isomorphism between LG and LLG gives rise to a canonical isomorphism between $\Omega[G]$ and $\Omega[LG]$.

Let $(\zeta_1, \dots, \zeta_n)$ be a basis of the space of invariant differential forms on S and let $(\xi_1, \dots, \xi_n)$ be the corresponding basis of the space of invariant differential forms on LS. Write $\omega_i(x, y) = \zeta_i(y) - \xi_i(x)$, for each $i = 1, \dots, n$.

Recall that the tangent bundle TS is identified with $LS \times S$, and Γ_S is defined by

$$(x, y) \in \Gamma_S \iff lD_{LG}(x) = lD_G(y).$$

Translating the definition of the logarithmic derivatives into coordinates using lemma 3.1 gives us an alternative characterization of Γ_S in terms of the differential forms ω_i.

Lemma 3.4. *Let $x \in LS(F)$ and $y \in S(F)$, and let the differentials ω_i be defined as above. Then $(x, y) \in \Gamma_S$ iff for each $i = 1, \dots, n$, the equation $D^*\omega_i(x, y) = 0$ holds.* $\square$

Consider a differential field of characteristic zero $\langle F; +, \cdot, D\rangle$, and let C be the constant field. As described in the introduction, we consider the reduct of F to the language $\langle F; +, \cdot, C, (\Gamma_S)_{S\in\mathcal{S}}, (\hat{c})_{c\in C_0}\rangle$. We also consider a slight generalization. Suppose now that F is a field with a family Δ of derivations, such that $C = \bigcap_{D\in\Delta} \ker D$. For each $D \in \Delta$, we can consider the solution set $\Gamma_{S,D}$ of the exponential differential equation for S with respect to D. Write $\Gamma_S = \bigcap_{D\in\Delta} \Gamma_{S,D}$.

Given a finite set of derivations $\Delta = \{D_1, \ldots, D_r\}$ on F, and a tuple $a = \langle a_1, \ldots, a_n\rangle$ from F, define the Jacobian matrix of a with respect to Δ to be

$$\mathrm{Jac}_\Delta(a) = \begin{pmatrix} D_1 a_1 & \cdots & D_1 a_n \\ \vdots & \ddots & \vdots \\ D_r a_1 & \cdots & D_r a_n \end{pmatrix}$$

and write $\mathrm{rk}\,\mathrm{Jac}_\Delta(a)$ to be the rank of this matrix.

If Δ is an infinite set of derivations, the rank of the Jacobian matrix is then defined to be

$$\mathrm{rk}\,\mathrm{Jac}_\Delta(a) = \max\left\{\mathrm{rk}\,\mathrm{Jac}_{\Delta'}(a) \mid \Delta' \text{ is a finite subset of } \Delta\right\}.$$

The rank of the matrix is bounded by the number n of columns, so this maximum is well defined. We will not usually write the dependence on Δ explicitly, so will write this simply as $\mathrm{rk}\,\mathrm{Jac}(a)$.

Proposition 3.5. *Let $\langle F; +, \cdot, D\rangle$ be a differential field, let C_0 be a subfield of the field of constants C, and let $\mathcal{S}$ be a collection of semiabelian varieties, each defined over C_0. Then the reduct $\langle F; +, \cdot, C, \{\hat{c}\}_{c\in C_0}, \{\Gamma_S\}_{S\in\mathcal{S}}\rangle$ satisfies the axioms U2—U7 and U1$'$, the universal part of U1.*

Proof. Axiom U1$'$ says that F is a field, C is a relatively algebraically closed subfield, and the constants $\hat{c}$ have the correct algebraic type, all of which holds in the reduct.

For U2, Γ_S is the kernel of the group homomorphism

$$\begin{array}{rcl} TS(F) & \longrightarrow & LS(F) \\ (x, y) & \longmapsto & lD_S(y) - lD_{LS}(x) \end{array}$$

and so is a subgroup of TS.

The logarithmic derivatives lD_S and lD_{LS} vanish on the C-points of S and LS respectively, so $TS(C) \subseteq \Gamma_S$, which is U3.

The fibre of $x = 0$ is $\{y \in S(F) \mid lD_S(y) = 0\}$ which is $S(C)$. Similarly, the fibre of $y = 0$ is $LS(C)$. This is axiom U4.

Suppose that $S_1 \xrightarrow{f} S_2$ is an algebraic group homomorphism, and let $(x, y) \in \Gamma_{S_1}$. Let ζ be an invariant differential form on S_2, let ξ be the corresponding invariant form on LS_2, and let $\omega = \zeta - \xi$. To show $Tf(x, y) \in \Gamma_{S_2}$, it suffices to show that $D^*\omega(Tf(x, y)) = 0$. But

$$D^*\omega(Tf(x, y)) = D^*\zeta(f(x)) - D^*\xi(df_e(y)) = D^*(f_*\zeta)(y) - D^*(df_{e*}(\xi))(x)$$

where f_* and df_{e*} denote the images of f and df_e under the contravariant cotangent bundle functor. The image of an invariant form is an invariant form, and so $f_*\zeta$ and $df_{e*}(\xi)$ are corresponding invariant differential forms on S_1 and LS_1. Hence $D^*\omega(Tf(x, y)) = 0$, since $(x, y) \in \Gamma_{S_1}$.

Now suppose that f is an isogeny. Let $(v, w) \in \Gamma_{S_2}$ and let $(x, y) \in TS_1$ such that $Tf(x, y) = (v, w)$. Let ζ be an invariant form on S_1, let ξ be the corresponding invariant form on LS_1, and let $\omega = \zeta - \xi$ on TS_1. Since f is an isogeny, the map Tf_* is an isomorphism between the spaces of invariant forms on TS_2 and TS_1. Let $\eta = (Tf_*)^{-1}(\omega)$. Now

$$D^*(\omega(x, y)) = D^*(Tf_*\eta)(x, y) = D^*\eta((Tf)(x, y)) = D^*\eta(v, w) = 0$$

so $(x, y) \in \Gamma_{S_1}$. This proves axiom U5.

If $S_1 \subseteq S_2$, let $\zeta_1, \ldots, \zeta_m$ be a basis of invariant differential forms on S_1 and extend to a basis $\zeta_1, \ldots, \zeta_n$ of invariant differential forms on S_2. Let $\xi_1, \ldots, \xi_n$ be the corresponding basis of invariant differential forms on LS_2, and let $\omega_i(x, y) = \zeta_i(y) - \xi_i(x)$. If $g \in \Gamma_{S_1}$ then $D^*\omega_i(g) = 0$ for $i = 1, \ldots, m$ by definition of Γ_{S_1} and for $i = m + 1, \ldots, n$ because $g \in TS_1$. So $g \in \Gamma_{S_2}$. Conversely, if $g \in \Gamma_{S_2} \cap TS_1$ then $D^*\omega_i(g) = 0$ for $i = 1, \ldots, m$ and so $g \in \Gamma_{S_1}$. So U6 holds.

The logarithmic derivative of a product is given componentwise, that is, $lD_{G_1 \times G_2}(g_1, g_2) = (lD_{G_1}(g_1), lD_{G_2}(g_2))$. Axiom U7 follows. $\square$

3.3 The Schanuel property

Next we prove the Schanuel property, in a slightly stronger form for differential fields with a family of derivations. The following lemma on algebraic subgroups of TS is central to the proof.

Lemma 3.6. *Let S be a semiabelian variety, and let G be an algebraic subgroup of $TS = LS \times S$. Then G is of the form $G_1 \times G_2$ for some subgroup G_1 of LS and some subgroup G_2 of S.*

Proof. Let $G_1 = \mathrm{pr}_{LS}(G)$ and $G_2 = \mathrm{pr}_S(G)$. Write 0 for the identity element of LS and 1 for the identity element of S. Define subgroups $H_1 = \{x \in G_1 \mid (x,1) \in G\}$ and $H_2 = \{y \in G_2 \mid (0,y) \in G\}$ and define a quotient map $G_2 \xrightarrow{\theta} G_1/H_1$ by $\theta(y) = \{x \in G_1 \mid (x,y) \in G\}$. It is easy to check that θ is a regular group homomorphism with kernel H_2.

G_1/H_1 is a vector group, since algebraic subgroups and quotients of vector groups are vector groups. G_2 is an algebraic subgroup of a semiabelian variety, so is semiabelian-by-finite. But the only regular homomorphism from a semiabelian-by-finite group to a vector group is the zero homomorphism, so $H_2 = G_2$, and thus also $H_1 = G_1$. Hence $G = G_1 \times G_2$. □

We separate out the following intermediate step from the proof of the Schanuel property, as it will also be used later to prove EC. Recall the definition of the ω_i from before lemma 3.4.

Proposition 3.7. *Suppose $(x,y) \in \Gamma_S$ and the differentials $\omega_1(x,y), \ldots, \omega_n(x,y)$ are F-linearly dependent in $\Omega(F/C)$. Then there is a proper algebraic subgroup H of S and a point $\gamma \in TS(C)$ such that (x,y) lies in the coset $\gamma \cdot TH$.*

Proof. ***Step 1: C-linear dependence***
Take $\alpha_i \in F$ such that $\sum_{i=1}^n \alpha_i \omega_i(x,y) = 0$ is a minimal F-linear dependence on the ω_i, that is, if $I = \{i \mid \alpha_i \neq 0\}$ then the F-linear dimension of $\{\omega_i \mid i \in I\}$ is $|I| - 1$. Dividing by some non-zero α_i, we may assume that for some $i = i_0$, $\alpha_{i_0} = 1$.

Applying the Lie derivative L_D for $D \in \Delta$ we get

$$\begin{aligned} 0 = L_D \sum_{i=1}^n \alpha_i \omega_i(x,y) &= \sum_{i=1}^n [(D\alpha_i)\omega_i(x,y) + \alpha_i L_D \omega_i(x,y)] \\ &= \sum_{i=1}^n [(D\alpha_i)\omega_i(x,y) + \alpha_i(dD^*\omega_i(x,y) + D^* d\omega_i(x,y))] \\ &= \sum_{i=1}^n (D\alpha_i)\omega_i(x,y) \end{aligned}$$

using the properties of the Lie derivative given in lemma 3.2. The last equality uses the fact that $(x, y) \in \Gamma_S$, and so $D^*\omega_i(x, y) = 0$ for each i. It also uses the fact that each differential $\omega_i(x, y)$ is a difference of invariant differentials, and hence is closed by lemma 3.3, so $d\omega_i(x, y) = 0$ for each i.

Now $\alpha_{i_0} = 1$, so $D\alpha_{i_0} = 0$ but then, by the minimality of set I, we have that $D\alpha_i = 0$ for every i and each $D \in \Delta$, so each $\alpha_i \in C$. Hence the $\omega_i(x, y)$ are C-linearly dependent.

Step 2:[1] ***A subgroup of*** TS

Let $\eta = \sum_{i=1}^{n} \alpha_i\omega_i$. Then η is an invariant differential form on TS, defined over C.

By lemma 3.1, η defines a group homomorphism $TS \longrightarrow \Omega(F/C)$, so $\ker\eta$ is a subgroup of TS. The ω_i are linearly independent, so $\eta \neq 0$ and hence $\ker\eta$ is a proper subgroup of TS. By construction, $(x, y) \in \ker\eta$.

Let $V = \mathrm{Loc}_C(x, y)$, the algebraic locus of (x, y) over C, and an algebraic subvariety of TS. The field C is algebraically closed, so V has a C-point, say $\gamma = (\gamma_1, \gamma_2)$, with $\gamma_1 \in LS$ and $\gamma_2 \in S$. Let $V' = \{v\gamma^{-1} \mid v \in V\}$. Then V' is an irreducible algebraic variety defined over C, containing the identity of TS, and having $(x', y') = (x\gamma_1^{-1}, y\gamma_2^{-1})$ as a generic point over C.

Let O be the orbit of (x', y') in the algebraic closure $\bar{F}$ of F, under $\mathrm{Aut}(\bar{F}/C)$, that is, automorphisms of the pure field.

For $n \in \mathbb{N}$, let $nV' = \{v_1 \cdot \dots \cdot v_n \mid v_i \in V'\}$, and similarly nO. By the indecomposability theorem due to Chevalley [Che51, Chapter II, section 7] (see also [Mar02, p261]), there is $n \in \mathbb{N}$ such that $nV' = G$, an algebraic subgroup of TS. Now $nO \subseteq nV'$ (where now we identify nV' with its $\bar{F}$-points) and O contains all realizations of the generic type of V', that is, of $\mathrm{tp}(x/C)$, hence nO contains all the realizations of the generic types of nV'. Every element of G is the product of two generic elements, so $2nO = G$.

The differential form η vanishes on $TS(C)$, so

$$\eta(x', y') = \eta(x, y) - \eta(\gamma_1, \gamma_2) = 0$$

but then η vanishes on O, because η is defined over C and hence its kernel is $\mathrm{Aut}(\bar{F}/C)$-invariant. Since η is a group homomorphism, it vanishes on the subgroup G generated by O, that is $G \subseteq \ker\eta$. Since $\ker\eta$ is a proper subgroup of TS, G is a proper algebraic subgroup of TS. By lemma 3.6, G is of the form $J \times H$, with J a subgroup of LS and H a subgroup of S.

[1] Thanks to Piotr Kowalski for an improved argument in step 2.

***Step 3: A subgroup of** S*
Recall that $\omega_i(x, y) = \zeta_i(y) - \xi_i(x)$, with the ζ_i being invariant forms on S and the ξ_i being invariant forms on LS. Let $\nu = \sum_{i=1}^n \alpha_i \zeta_i$ and $\mu = \sum_{i=1}^n \alpha_i \xi_i$.

For any $h \in H$,

$$\nu(h) = \nu(h) - \mu(0) = \eta(0, h) = 0$$

because $(0, h) \in G \subseteq \ker \eta$. Thus $H \subseteq \ker \nu$. Now ν is a nonzero invariant form on S, since the ζ_i are linearly independent. Hence H is a proper algebraic subgroup of S.

Step 4: Constant cosets
Consider the quotient group $\Gamma_S / TS(C)$. By axiom U4, it is the graph of a bijection

$$\frac{\mathrm{pr}_{LS} \Gamma_S}{LS(C)} \xrightarrow{\theta} \frac{\mathrm{pr}_S \Gamma_S}{S(C)}$$

where pr_{LS} is the projection $TS \longrightarrow LS$ and pr_S is the projection $TS \longrightarrow S$. By the choice of corresponding bases of invariant forms $\zeta_1, \ldots, \zeta_n$ on S and $\xi_1, \ldots, \xi_n$ on LS, and lemma 3.4,

$$\theta^{-1}((\mathrm{pr}_1 \Gamma_S \cap H) \cdot S(C)) = (\mathrm{pr}_2 \Gamma_S \cap LH) \cdot LS(C)$$

By construction of H, y lies in a constant coset of H, and $(x, y) \in \Gamma_S$, so $\theta^{-1}(y \cdot S(C)) = x \cdot LS(C)$, hence x lies in a constant coset of LH. Thus (x, y) lies in a constant coset of TH, as required. □

Theorem 3.8 (The Schanuel property)**.** *Let F be a field of characteristic zero, let Δ be a collection of derivations on F, and let C be the intersection of their constant fields. Let S be a semiabelian variety defined over C, of dimension n, and let $\Gamma_S \subseteq LS \times S$ be the solution set of the exponential differential equation of S (that is, the intersection of the solution sets for each $D \in \Delta$).*

Suppose that $(x, y) \in \Gamma_S$ and $\mathrm{td}(x, y/C) - \mathrm{rk}\,\mathrm{Jac}(x, y) < n$. Then there is a proper algebraic subgroup H of S and a constant point $\gamma \in TS(C)$ such that (x, y) lies in the coset $\gamma \cdot TH$.

Proof. To prove the theorem, it suffices by proposition 3.7 to show that the differential forms $\omega_1(x, y), \ldots, \omega_n(x, y)$ are F-linearly dependent in $\Omega(F/C)$.

Let $E = C(x, y)$, the subfield (not differential subfield) of F generated over C by x and y. Choose a finite tuple $D_1, \ldots, D_r$ of derivations

from Δ such that the rank of the Jacobian matrix $\operatorname{rk}\operatorname{Jac}_{\Delta}(x,y)$ is equal to $\operatorname{rk}\operatorname{Jac}_{D_1,\ldots,D_r}(x,y)$. Write D for the tuple $(D_1,\ldots,D_r)$, a map $F \xrightarrow{D} F^r$. Consider the diagram below, where D^* is the F-linear map which comes from the universal property of d.

$$\begin{tikzcd} F \arrow[rr, "d"] \arrow[drr, "D"'] & & \Omega(F/C) \arrow[d, "D^*"] \\ & & F^r \end{tikzcd}$$

Write $\operatorname{Ann}(D)$ for the kernel of the linear map D^*. The diagram restricts to

$$\begin{tikzcd} E \arrow[rr, "d"] \arrow[drr, "D"'] & & \Omega(E/C)\otimes_E F \arrow[d, "D^*"] \\ & & F^r \end{tikzcd}$$

where again D^* is F-linear, with kernel $(\Omega(E/C)\otimes_E F)\cap\operatorname{Ann}(D)$.

The E-vector space $\Omega(E/C)$ has E-linear dimension equal to $\operatorname{td}(x,y/C)$, and so $\Omega(E/C)\otimes_E F$ has F-linear dimension also equal to $\operatorname{td}(x,y/C)$.

The image of D^* is the image of D, which is spanned by the columns of the matrix $\operatorname{Jac}(x,y)$. Thus $\operatorname{rk}\operatorname{Jac}(x,y)$ is equal to the rank of the linear map D^*, which by the rank-nullity theorem is equal to the codimension of its kernel. Thus $\Omega(E/C)\otimes_E F\cap\operatorname{Ann}(D)$ has dimension $\operatorname{td}(x,y/C)-\operatorname{rk}\operatorname{Jac}(x,y)$, which by assumption is strictly less than n.

The differential forms ω_i are defined over C, so each of the n differentials $\omega_i(x,y)$ lies in $\Omega(E/C)$. Since $(x,y)\in\Gamma_S$, each $\omega_i(x,y)$ also lies in $\operatorname{Ann}(D)$. Hence they are E-linearly dependent, and in particular they are F-linearly dependent. □

Corollary 3.9. *The reduct of a differential field to the language $\mathcal{L}_{\mathcal{S}}$ satisfies the SP axiom.*

Proof. The axiom SP is just the special case of theorem 3.8 for the semiabelian varieties which lie in $\mathcal{S}$, with Δ being the singleton $\{D\}$. □

3.4 Existential closedness

Theorem 3.10. *Let F be a differentially closed field (of characteristic zero, with one derivation). Then the reduct of F to the language $\mathcal{L}_{\mathcal{S}}$ has the EC property.*

Proof. Let $S \in \mathcal{S}$, let $n = \dim S$, and let V be a perfectly rotund subvariety of TS, defined over F. Let $(W_e)_{e \in Q(C)}$ be a parametric family of proper subvarieties of V, defined over C_0. We show there is $(x, y) \in \Gamma_S \cap V \smallsetminus \bigcup_{e \in Q(C)} W_e$. By proposition 2.33, this suffices to prove the EC property.

Let D_0 be the derivation on F. Let (x, y) be a generic point of V over F, and let $K = F(x, y)^{\mathrm{alg}}$, the algebraic closure of $F(x, y)$.

We wish to consider the derivations in $\mathrm{Der}(K/C)$ which extend D_0 on F. These form a coset of the subspace $\mathrm{Der}(K/F)$ of $\mathrm{Der}(K/C)$. In order to work with subspaces rather than cosets, we follow [Pie03] in defining

$$\mathrm{Der}(K/D_0) = \{D \in \mathrm{Der}(K/C) \mid \exists \lambda \in K, D\restriction_F = \lambda D_0\}$$

which can be considered as the dual space of a quotient $\Omega(K/D_0)$ of $\Omega(K/C)$. This gives a sequence of inclusions

$$\mathrm{Der}(K/F) \hookrightarrow \mathrm{Der}(K/D_0) \hookrightarrow \mathrm{Der}(K/C)$$

and dually surjections

$$\Omega(K/C) \twoheadrightarrow \Omega(K/D_0) \twoheadrightarrow \Omega(K/F)$$

of K-vector spaces.

We can consider the differentials $\omega_i(x, y)$ in $\Omega(K/C)$, and also in $\Omega(K/D_0)$ and $\Omega(K/F)$ via the canonical surjections above. By the rotundity of V and the genericity of (x, y) in V over F, (x, y) does not lie in an F-coset of TH for any proper algebraic subgroup H of S. Hence, by the contrapositive of proposition 3.7, the differentials $\omega_1(x, y), \ldots, \omega_n(x, y)$ are K-linearly independent in $\Omega(K/F)$, and hence also in $\Omega(K/D_0)$ and $\Omega(K/C)$.

The K-linear dimension of $\Omega(K/D_0)$ is equal to that of $\mathrm{Der}(K/D_0)$, which is $\dim \mathrm{Der}(K/F) + 1$, the "+1" because $F \neq C$. As V is perfectly rotund it has dimension n and, because (x, y) is a generic point of V over F and $K = F(x, y)$, we have $\dim \mathrm{Der}(K/F) = n$.

Let $\Lambda = \langle \omega_1(x, y), \ldots, \omega_n(x, y) \rangle$ be the span of the $\omega_i(x, y)$ in $\Omega(K/C)$, with annihilator $\mathrm{Ann}(\Lambda) \subseteq \mathrm{Der}(K/C)$. The image of Λ has codimension 1 in

$\Omega(K/D_0)$, so $\mathrm{Der}(K/D_0) \cap \mathrm{Ann}(\Lambda)$ has dimension 1. Let $D \in \mathrm{Der}(K/D_0) \cap \mathrm{Ann}(\Lambda)$ be nonzero. The image of Λ spans $\Omega(K/F)$, so $\mathrm{Der}(K/F) \cap \mathrm{Ann}(\Lambda) = \{0\}$. Hence $D\restriction_F = \lambda D_0$ for some non-zero λ. Replacing D by $\lambda^{-1}D$, we may assume that $\lambda = 1$, that is, D extends D_0. Indeed, we have shown that this D is the unique derivation on K extending D_0 such that $(x, y) \in \Gamma_S$ with respect to D.

Let K' be the differential closure of $\langle K; D\rangle$, and let C_K be the field of constants in K'. Since K is algebraically closed, $C_K \subseteq K$. We must show that $C_K = C$. Let F' be the algebraic closure in K' of $C_K \cup F$. Now $F \subseteq K'$ is an inclusion of differentially closed fields, and the theory DCF_0 has quantifier elimination, so the inclusion is an elementary inclusion. Thus it preserves all formulas in the differential field language, and in particular all existential formulas in the language $\mathcal{L}_{\mathcal{S}}$. It follows that it is a strong embedding when considered as an embedding of the reducts to the language $\mathcal{L}_{\mathcal{S}}$. So $F \lhd K'$, and hence $F \lhd K$. Furthermore, $F' \lhd K$ since F' is obtained from F just by adding new constants. Thus $\delta(x, y/F') \geqslant 0$. Let H be the smallest algebraic subgroup such that (x, y) lies in a C_K-coset of TH, say $\gamma \cdot TH$. Then

$$\dim H \leqslant \mathrm{td}(x, y/F') \leqslant \dim(V \cap \gamma \cdot TH)$$

because $F' \lhd K$, and because $(x, y) \in V \cap \gamma \cdot TH$ which is defined over F'. But V is perfectly rotund, so $\dim H > \dim(V \cap \gamma \cdot TH)$ unless $H = S$. Thus $\mathrm{td}(x, y/F') = n = \mathrm{td}(x, y/F)$, so $F = F'$ and $C_K = C$.

Now (x, y) is generic in V over C, which means that it does not lie in any proper subvariety of V defined over C. Thus we have

$$K \models (\exists (x, y) \in TS)(\forall e \in Q(C))[(x, y) \in \Gamma_S \cap V \wedge (x, y) \notin W_e]$$

and this sentence remains true in K' because there are no new constants. Since F is an elementary substructure of K', it also satisfies the same sentence. Thus F satisfies the EC property. $\square$

We can now give criteria for a system of exponential differential equations to have a solution in some differential field. The Schanuel property can be viewed as a necessary condition for a system of differential equations to have a solution, and the EC property gives a matching sufficient condition.

Let F be a differentially closed field, let S be a semiabelian variety defined over the constant subfield C, and let V be a subvariety of TS. Firstly, we replace $V \subseteq TS$ by a homomorphic image $V' \subseteq TS'$ which is free, with

$\mathrm{Loc}_C V'$ absolutely free. If V' is defined over C then a necessary and sufficient condition for there to be a nonconstant point in $\Gamma_{S'} \cap V'$ in F is for V' to be strongly rotund.

If V' is not defined over C then a sufficient condition for a point to exist is for V' to be rotund. If in addition $\mathrm{Loc}_C V'$ is strongly rotund then a nonconstant point exists. Any such point gives rise to a point in $\Gamma_S \cap V$ by taking an inverse image under the quotient map.

The reduct of a differentially closed field does not have quantifier elimination in the language $\mathcal{L}_S$, so there is no general necessary and sufficient condition when V is defined with non-constant parameters. The theory DCF_0 does have quantifier elimination, so there must be a condition which depends on what other differential equations the parameters satisfy.

4 The first order theory

4.1 The uniform Schanuel property

The compactness theorem of first order model theory can be combined with the Schanuel property to give a *uniform* Schanuel property.

The algebraic subgroups of ${\mathbb{G}_\mathrm{a}}^n$ are uniformly definable by formulas of the form $Mx = 0$, where M ranges over the definable set of matrices $\mathrm{Mat}_{n\times n}$. In other words, the algebraic subgroups form a parametric family in the sense of definition 2.29. However, for all other commutative algebraic groups the set of all algebraic subgroups is not uniformly definable, and for semiabelian varieties there are no infinite parametric families of algebraic subgroups at all. This lack of uniform definability in fact works in our favour.

We use the fibre condition of algebraic geometry, from [Sha94, page 77].

Lemma 4.1 (Fibre Condition). *Let $(V_p)_{p\in P}$ be a family of algebraic varieties, parametrized over a constructible set P. Then for each $k \in \mathbb{N}$, the set of fibres $\{p \in P \mid \dim V_p \geqslant k\}$ is a subvariety of P and the set $\{p \in P \mid \dim V_p = k\}$ is constructible.* □

A similar result holds for the rank of the Jacobian matrix in a differential field with finitely many commuting derivations. Indeed, upon close examination the main part of the proof of the fibre condition is more or less this result.

Lemma 4.2. *For each algebraic variety V and for each natural number k, the set $\{x \in V \mid \operatorname{rk}\operatorname{Jac}(x) \leqslant k\}$ is positively definable in the language of differential fields, and the set $\{x \in V \mid \operatorname{rk}\operatorname{Jac}(x) = k\}$ is definable.*

Proof. V is made up of finitely many affine charts, so it is enough to consider V to be affine. For each x the Jacobian $\operatorname{Jac}(x)$ is an $r \times n$ matrix. Its rank is the largest k such that there is a $k \times k$ minor matrix with non-zero determinant. Thus $\operatorname{rk}\operatorname{Jac}(x) \leqslant k$ iff $\det M = 0$ for every minor matrix M of size $k+1$. The determinant is a polynomial and there are only finitely many minors, so this finite conjunction of equations is a positive first order condition on a matrix in the field language. The entries in the Jacobian are terms in the differential field language, and so we have positive definability of $\operatorname{rk}\operatorname{Jac}(x) \leqslant k$. The second part follows. $\square$

Theorem 4.3 (Uniform Schanuel property)**.** *Let F be a differential field of characteristic zero, with finitely many commuting derivations. Let S be a semiabelian variety of dimension n, defined over the constant subfield C of F. For each parametric family $(V_c)_{c \in P(C)}$ of subvarieties of TS, with V_c defined over $\mathbb{Q}(c)$, there is a finite set $\mathcal{H}_V$ of proper algebraic subgroups of S such that for each $c \in P(C)$ and each $(x, y) \in \Gamma_S \cap V_c$, if $\dim V_c - \operatorname{rk}\operatorname{Jac}(x, y) = n - t$ with $t > 0$, then there is $\gamma \in TS(C)$ and $H \in \mathcal{H}_V$ of codimension at least t in S such that (x, y) lies in the coset $\gamma \cdot TH$.*

Proof. The set

$$\Phi_V = \{((x, y), c) \in \Gamma_S \times P(C) \mid (x, y) \in V_c, \dim V_c - \operatorname{rk}\operatorname{Jac}(x, y) = n - t\}$$

is definable using lemmas 4.1 and 4.2. The set of formulas

$$((x, y), c) \in \Phi_V \wedge (\exists \gamma \in TS(C))[(x, y) \in \gamma \cdot TH]$$

where H ranges over all proper algebraic subgroups of S of codimension at least t is countable (as there are only countably many proper algebraic subgroups of S); in particular it is of bounded size. It is unsatisfiable by the Schanuel property, so by the compactness theorem some finite subset of it is unsatisfiable. This gives the finite set $\mathcal{H}_V$. $\square$

For definiteness, we choose $\mathcal{H}_V$ to be a particular minimal finite set of subgroups for each variety V. The compactness method gives no information about the nature of $\mathcal{H}_V$, beyond it being finite.

Corollary 4.4. *The SP axiom can be written as a first order axiom scheme in the language $\mathcal{L}_S$.*

Proof. For each variety P and each parametric family $(V_p)_{p \in P}$ of algebraic subvarieties of TS, defined over $\mathbb{Q}$, take the axiom

$$(\forall p \in P(C))(\forall g \in \Gamma_S \cap V_p) \left[\dim V_p \leqslant \dim S \to \bigvee_{H \in \mathcal{H}_V} q_H(g) \in T(S/H)(C) \right]$$

where $\mathcal{H}_V$ is the finite set of algebraic subgroups of S given by theorem 4.3 and q_H is the quotient map $TS \xrightarrow{q_H} T(S/H)$. □

4.2 The Weak CIT

We next give a purely algebraic result about the intersection of subvarieties and algebraic subgroups of a semiabelian variety. The proof here is in essence the same as the proof of Zilber, but simplified by using the full Schanuel property for partial differential fields rather than just ordinary differential fields, and by separating off the statement and proof of the uniform Schanuel property.

Definition 4.5. Let U be a smooth irreducible algebraic variety, and let V, W be subvarieties of U, with $V \cap W \neq \emptyset$. The intersection $V \cap W$ is said to be *typical* (in U) iff

$$\dim(V \cap W) = \dim V + \dim W - \dim U$$

and *atypical* iff

$$\dim(V \cap W) > \dim V + \dim W - \dim U.$$

Even if V and W are irreducible, the intersection $V \cap W$ may be reducible, and its components may have different dimensions. We say that a component X of $V \cap W$ is *atypical* iff

$$\dim X > \dim V + \dim W - \dim U.$$

We also say that the *degree of atypicality* is the difference

$$\dim X - (\dim V + \dim W - \dim U).$$

Note that the intersection is typical iff $\operatorname{codim}(V \cap W) = \operatorname{codim} V + \operatorname{codim} W$, and since U is smooth the dimension of the intersection cannot be less than the typical size (assuming the intersection is nonempty).

Theorem 4.6 ("Weak CIT" for semiabelian varieties)**.** *Let S be a semiabelian variety defined over an algebraically closed field C of characteristic zero. Let $(U_p)_{p \in P}$ be a parametric family of algebraic subvarieties of S. There is a finite family $\mathcal{J}_U$ of proper algebraic subgroups of S such that, for any coset $\kappa = a \cdot H$ of any algebraic subgroup H of S and any $p \in P(C)$, if X is an atypical component of $U_p \cap \kappa$ with degree of atypicality t, then there is $J \in \mathcal{J}_U$ of codimension at least t and $s \in S(C)$ such that $X \subseteq s \cdot J$.*

Furthermore, we may assume that X is a typical component of the intersection $(U_p \cap s \cdot J) \cap (\kappa \cap s \cdot J)$ in $s \cdot J$.

The weak CIT is a simple corollary of the uniform Schanuel property, but as well as the fact that there are no parametric families of subgroups of a semiabelian variety, we use the fact that the subgroups of a vector group *do* form a parametric family.

Proof. Let $n = \dim S$ and define $\Lambda_{Ma} = \{x \in LS \mid Mx = a\}$ where M is an $n \times n$ matrix and $a \in LS$. So Λ is the parametric family of all cosets of algebraic subgroups of LS.

Suppose that X is an atypical component of $U_p \cap \kappa$ with

$$r = \dim X = (\dim U_p + \dim \kappa - \dim S) + t.$$

Let y be generic in X over C and let $D_1, \ldots, D_r$ be a basis of $\operatorname{Der}(C(y)/C)$. Then $\operatorname{rk} \operatorname{Jac}(y) = r$. Take $x \in LS(F)$ with F some differential field extension such that $(x, y) \in \Gamma_S$. Then $\operatorname{rk} \operatorname{Jac}(x, y) = \operatorname{rk} \operatorname{Jac}(y)$. Now $y \in \kappa$, a constant coset of the algebraic subgroup H of S, so, by axiom U4 (see also step 4 of the proof of proposition 3.7), x lies in a constant coset of LH. Thus x lies in Λ_{Ma} for a suitable choice of $M \in \operatorname{Mat}_{n \times n}(C)$ and $a \in LS(C)$, with $\dim \Lambda_{Ma} = \dim \kappa$. Let $V_{Ma,p} = \Lambda_{Ma} \times U_p$. Then $(x, y) \in \Gamma_S \cap V_{Ma,p}$ and

$$\dim V_{Ma,p} - \operatorname{rk} \operatorname{Jac}(x, y) = \dim \kappa + \dim U_p - \dim X = \dim S - t$$

and so by theorem 4.3, there is $s \in S(C)$ and an algebraic subgroup J of S of codimension at least t from the finite set $\mathcal{H}_V$ such that $y \in s \cdot J$. Since y is generic in X over C and $s \cdot J$ is defined over C, we have $X \subseteq s \cdot J$. Thus, in the notation of theorem 4.3, we may take the finite set $\mathcal{J}_U$ to be $\mathcal{H}_{\Lambda \times U}$.

Now $X \subseteq s \cdot J$, so $s^{-1} \cdot X \subseteq J$. Thus X is an atypical component of the intersection $(U_p \cap s \cdot J) \cap (\kappa \cap s \cdot J)$ in $s \cdot J$ iff $s^{-1} \cdot X$ is an atypical component of the intersection $(s^{-1} \cdot U_p \cap J) \cap (s^{-1} \cdot \kappa \cap J)$ in J. If so, we may inductively find a smaller subgroup $J' \subseteq J$ from a finite set and a point $s' \in J(C)$ such that $X \subseteq s' \cdot J'$. Thus, inductively, we may assume that X is a typical component of $(U_p \cap s \cdot J) \cap (\kappa \cap s \cdot J)$ in $s \cdot J$. $\square$

The special case of the theorem where S is an algebraic torus can be restated in more elementary, less geometric terms.

Corollary 4.7. *For each $n, d, r \in \mathbb{N}$, there is $N \in \mathbb{N}$ with the following property. Suppose that $x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n$ lies in an algebraic variety U defined by r polynomials of degree at most d, with coefficients in a subfield K of $\mathbb{C}$. Suppose also that x satisfies l multiplicative dependencies of the form $\prod_{i=1}^n x_i^{m_{ij}} = a_j$ with the $m_{ij} \in \mathbb{Z}$ and $a_j \in K$, and that $\mathrm{td}(K(x_1, \ldots, x_n)/K) = \dim U - l + t$, with $t > 0$.*

Then x satisfies t multiplicative dependencies with the powers m_{ij} having modulus at most N and the a_j lying in $\bar{K}$.

Proof. The subvarieties U of ${\mathbb{G}_\mathrm{m}}^n$ defined by r polynomials of degree at most d can be put into a single parametric family. Take $C = \bar{K}$ in 4.6. $\square$

This statement for tori has independently been reproved by Bombieri, Masser, and Zannier in [BMZ07]. They also use Ax's theorem (the Schanuel property for the exponential equation) but use a heights argument rather than the compactness theorem to get the natural number N. This gives them an explicit bound which cannot be obtained directly from the compactness theorem. Masser has noted in a private communication to me that their method should also extend to the semiabelian case.

4.3 Definability of rotundity

We generalize and adapt the proof in section 3 of [Zil05] to show that rotundity is a definable property of a variety. As well as the notion of an atypical intersection, we also need the notion of an atypical image of a variety under a map, in the context of subvarieties of groups.

Definition 4.8. Let G be an algebraic group, H an algebraic subgroup and V an algebraic subvariety of G. Let $G \xrightarrow{q} G/H$ be the quotient map onto

the coset space and write V/H for the image of V under q. This image V/H is said to be *typical* iff

$$\dim V/H = \min\{\dim G/H, \dim V\}$$

and *atypical* iff

$$\dim V/H < \min\{\dim G/H, \dim V\}.$$

We use the fact that in the conclusion of theorem 4.6, X is a typical component of the intersection $(U_p \cap s\cdot H) \cap (\kappa \cap s\cdot H)$ in $s\cdot H$. For convenience we also choose the finite set $\mathcal{J}_W$ of subgroups of S given in the conclusion of that theorem to contain the trivial subgroup. The additive formula for fibres is used several times:

(AF) For an irreducible variety A and a surjective map $A \xrightarrow{f} B$,

$$\dim A = \dim B + \min_{b\in B} \dim f^{-1}(b).$$

Theorem 4.9. *Let S be a semiabelian variety and $V \subseteq TS$ an irreducible subvariety. If V is not rotund then there is $J \in \mathcal{J}_W$ where $W = \mathrm{pr}_S V$ such that $\dim V/TJ < \dim S/J$. That is, failure of rotundity is witnessed by a member of the finite set $\mathcal{J}_W$.*

Proof. Suppose that $\dim V/TH < \dim S/H$ for some algebraic subgroup H of S. If $H = 1$ is the trivial subgroup then we are done since $1 \in \mathcal{J}_W$, so we assume that $\dim V \geqslant \dim S$, and $H \neq 1$.

Step 1 The image W/H is atypical.

W/H is a projection of V/TH, so

$$\dim W/H \leqslant \dim V/TH < \dim S/H.$$

Thus if W/H were typical we would have $\dim W/H = \dim W$, so the fibres of the map $W \longrightarrow W/H$ would be finite. The fibres of $V \longrightarrow V/TH$ could then have dimension at most $\dim H$, so

$$\dim V/TH \geqslant \dim V - \dim H \geqslant \dim S - \dim H = \dim S/H$$

which contradicts the assumption. Thus W/H is atypical.

Step 2 There is $J \in \mathcal{J}_W$ such that

$$\dim W/J = \dim W/H - \dim J/(J \cap H) \tag{3}$$

and

$$\dim W/H = \dim W/(J \cap H). \tag{4}$$

Let $x \in W$ be generic over a field of definition of S, H and W, and let κ be the coset $x \cdot H$. Then $W \cap \kappa$ is a generic fibre of the quotient map so, by the addition formula for fibres (AF),

$$\dim W \cap \kappa = \dim W - \dim W/H$$

which is strictly positive as the image is atypical. Let X be the component of $W \cap \kappa$ containing x, which must be of maximal dimension by genericity of x. Thus

$$\dim X = \dim(W \cap \kappa) = \dim W - \dim W/H \tag{5}$$

and by atypicality of the image

$$\dim W/H < \dim S/H = \dim S - \dim H$$

so

$$\dim X > \dim W + \dim H - \dim S.$$

Now $\dim H = \dim \kappa$ so X is an atypical component of the intersection $W \cap \kappa$ in S. By theorem 4.6 there is $J \in \mathcal{J}_W$ such that X is contained in the coset $\kappa' = x \cdot J$. Thus the quotient of X by $J \cap H$ is isomorphic to the quotient by H, so since X is a component of maximal dimension this implies (4).

By the remark above, X is a typical component of $(W \cap \kappa') \cap (\kappa \cap \kappa')$ in κ', that is

$$\dim X = \dim(W \cap \kappa') + \dim(\kappa \cap \kappa') - \dim \kappa'. \tag{6}$$

Let Y be the connected component of $(W \cap \kappa')$ containing X. Then (6) becomes

$$\dim X = \dim Y + \dim(J \cap H) - \dim J. \tag{7}$$

Y is a generic fibre of $W \longrightarrow W/J$, so by (AF) again,

$$\dim Y = \dim W - \dim J. \tag{8}$$

Substituting (5) and (8) into (7) gives (3) as required.

Let $H' = J \cap H$.

Step 3 $\dim V/TH' < \dim S/H'$.

For $b \in W$ write $V_b \subseteq LS$ for the fibre of the projection $V \longrightarrow W$. The projection $LS/LH' \longrightarrow LS/LH$ has fibres of dimension $k = \dim S/H' - \dim S/H$, so for any b the fibres of the map $V_b/LH' \longrightarrow V_b/LH$ have dimension at most k. Thus

$$\dim V_b/LH' \leqslant \dim V_b/LH + k. \tag{9}$$

By (AF),

$$\dim V/TH' = \dim W/H' + \min_{b\in W} \dim V_b/LH' \tag{10}$$

and substituting in (10) using (4) and (9) gives

$$\dim V/TH' \leqslant \dim W/H + \min_{b\in W} \dim V_b/LH + k$$

which by (AF) again implies

$$\dim V/TH' \leqslant \dim V/TH + k < \dim S/H'$$

as required.

Step 4 $\dim V/TJ < \dim S/J$.

This is very similar to step 3. Since $H' \subseteq J$, the quotient factors as

$$V \longrightarrow V/TH' \longrightarrow V/TJ$$

so for any $b \in W$,

$$\dim V_b/LJ \leqslant \dim V_b/LH'. \tag{11}$$

By (AF),

$$\dim V/TJ = \dim W/J + \min_{b\in W} \dim V_b/LJ \tag{12}$$

and using (3) and (11) this becomes

$$\dim V/TJ \leqslant \dim W/H' + \min_{b\in W} \dim V_b/LH' + (\dim S/J - \dim S/H').$$

Applying (AF) a final time with the conclusion of Step 3 gives

$$\dim V/TJ < \dim S/J$$

as required. □

Corollary 4.10. *The EC axiom can be written as a first order axiom scheme in the language $\mathcal{L}_{\mathcal{S}}$.*

Proof. For a parametric family $(V_p)_{p\in P}$ of subvarieties of TS, let $\mathrm{Rot}_V(p)$ be given by

$$V_p \text{ is irreducible } \ \& \ \dim V_p = \dim S \ \& \ \bigwedge\nolimits_{J\in\mathcal{J}_{\mathrm{pr}_S V}} \dim V_p/TJ \geqslant \dim S/J.$$

By theorem 4.9, this says that V_p is rotund, irreducible, and of dimension $n = \dim S$. By [Hru92, lemma 3], for any parametric family of varieties $(V_p)_{p\in P}$ there is a first order formula in p expressing that V_p is irreducible. Hence by lemma 4.1 and the finiteness of $\mathcal{J}_{\mathrm{pr}_S V}$, there is a first-order formula in the language of fields expressing $\mathrm{Rot}_V(p)$.

By proposition 2.33, EC and EC$'$ are equivalent, so in the statement of EC it is enough to consider perfectly rotund subvarieties. In fact perfect rotundity is not definable, but every perfectly rotund subvariety is irreducible and of dimension n, so it is enough to consider just these subvarieties.

For each $S \in \mathcal{S}$ and each pair of parametric families $(V_p)_{p\in P}$, $(W_e)_{e\in Q(C)}$ of subvarieties of TS, the families defined over C_0, take the following axiom.

$$\begin{aligned}&(\forall p \in P)(\exists g \in TS)(\forall e \in Q(C))[\mathrm{Rot}_V(p) \to \\ &\qquad [g \in \Gamma_S \cap V_p \wedge (g \notin W_e \vee \dim W_e \cap V_p = \dim V_p)]]\end{aligned}$$

For the V_p which are irreducible, the last clause says that g does not lie in any of the W_e whose intersection with V_p is a proper subvariety of V_p. Hence this scheme of first-order sentences captures the EC property. ☐

4.4 The first order theory

Recall that $T_{\mathcal{S}}$ is the $\mathcal{L}_{\mathcal{S}}$-theory axiomatized by the algebraic axioms U1 — U7 and the Schanuel property SP, which are given on page 7, together with the existential closedness axiom EC and non-triviality NT, which are given on page 24.

Theorem 4.11. *For each set $\mathcal{S}$ of semiabelian varieties, the theory $T_{\mathcal{S}}$ is the complete first order theory of the reduct to the language $\mathcal{L}_{\mathcal{S}}$ of a differentially closed field.*

Proof. We have shown that axioms U1 — U7 are first order in lemma 2.1, that the Schanuel property is first order in corollary 4.4, and that existential closedness is first order in corollary 4.10. It is immediate that NT is a first order axiom. Hence $T_{\mathcal{S}}$ is a first order theory. Proposition 3.5 shows that the reduct satisfies U1 — U7, corollary 3.9 says that it satisfies SP, and theorem 3.10 says that it satisfies EC. NT is immediate.

Since $T_{\mathcal{S}}$ is a first order theory, the part of proposition 2.21 which states that $\mathcal{K}^{\lhd}_{<\aleph_0}$ has only countably many objects and countably many extensions of each object shows that every completion of $T_{\mathcal{S}}$ is $\aleph_0$-stable, and so (since it has no finite models) has a countable saturated model. Let M be a countable saturated model of $T_{\mathcal{S}}$. By saturation, $\mathrm{td}(C/C_0) = \aleph_0$. For each $n \in \mathbb{N}$, there is a unique n-type of a cl-independent n-tuple. All of these types are realised in M, and hence M satisfies ID. We claim that M satisfies SEC.

Let $S \in \mathcal{S}$, let $V \subseteq TS$ be a rotund subvariety, and $A \subseteq F$ a finitely generated field of definition of V. For each proper subvariety W of V, defined over A, we may use the Rabinovich trick to replace $V \smallsetminus W$ by a some $V' \subseteq TS'$ for some larger $S' \in \mathcal{S}$ as follows. Let $\bar{x}$ be the coordinates (homogeneous coordinates if necessary) of the variety TS, and say that W is given by the equations $f_i(\bar{x}) = 0$ for $i = 1, \ldots, m$. Let $S_1, \ldots, S_m \in \mathcal{S}$ be nontrivial, and for each i let z_i be a coordinate of the Lie algebra LS_i. Let $S' = S \times \prod_{i=1}^{m} S_i$ and let V' be the subvariety of TS' given by $\bar{x} \in V$ and the equations $f_i(\bar{x})z_i = 1$ for $i = 1, \ldots, m$. Then V' is a rotund subvariety of TS'.

If necessary, we may now intersect V' with generic hyperplanes as in the proof of proposition 2.33 to ensure that $\dim V' = \dim S'$. We can regard a family $(W_e)_{e \in Q(C)}$ of proper subvarieties of V, defined over C, as a family of subvarieties of V' via the obvious co-ordinate maps. Now by EC there is $h \in \Gamma_{S'} \cap V' \smallsetminus \bigcup_{e \in Q(C)} W_e$. By the definition of V', the projection g of h to TS lies in $\Gamma_S \cap (V \smallsetminus W \smallsetminus \bigcup_{e \in Q(C)} W_e$. Hence, by the $\aleph_0$-saturation of M, there is $g \in \Gamma_S \cap V$, generic in V over $A \cup C$. That is, SEC holds in M.

Thus, by theorem 2.35, M is isomorphic to the Fraissé limit U. So $T_{\mathcal{S}}$ has exactly one countable saturated model, so only one completion, and hence it is complete. □

We end with two simple observations about the theories $T_{\mathcal{S}}$.

Proposition 4.12. *For each set $\mathcal{S}$, the theory $T_{\mathcal{S}}$ has Morley rank ω.*

Proof. $T_{\mathcal{S}}$ is a reduct of DCF_0, hence it has Morley rank at most ω. It has the theory of pairs of algebraically closed fields as a reduct (which in fact is

$T_{\mathcal{S}}$ for $\mathcal{S} = \{1\}$), which has Morley rank ω, so $T_{\mathcal{S}}$ has Morley rank ω. $\square$

Proposition 4.13. *If $\mathcal{S}$ and $\mathcal{S}'$ are distinct collections of semiabelian varieties, each closed under products, subgroups, quotients, and under isogeny, then $T_{\mathcal{S}} \neq T_{\mathcal{S}'}$. (They are theories in different languages, so we mean they have no common definitional expansion.) Furthermore, all the theories $T_{\mathcal{S}}$ are proper reducts of (expansions by constant symbols of)* DFC_0.

Proof. Let F be an $\aleph_0$-saturated differentially closed field. Without loss of generality $\mathcal{S} \nsubseteq \mathcal{S}'$, so take $S \in \mathcal{S} \smallsetminus \mathcal{S}'$. Choose an absolutely free and strongly rotund subvariety V of TS, of dimension $\dim S + 1$. Then F contains a point $g \in \Gamma_S \cap V$ with $\mathrm{grk}(g) = \dim S$, by SEC for the reduct of F to $\mathcal{L}_{\mathcal{S}}$.

By the Schanuel property, g cannot be algebraically dependent on any point $h \in \Gamma_{S'}$ for any $S' \in \mathcal{S}'$. Hence g has dimension $\dim S + 1$ in the sense of the pregeometry of the reduct to $\mathcal{L}_{\mathcal{S}'}$, but only dimension 1 in the sense of the pregeometry of the reduct to $\mathcal{L}_{\mathcal{S}}$. Thus the theories $T_{\mathcal{S}}$ and $T_{\mathcal{S}'}$ are distinct reducts of DCF_0. Since every set $\mathcal{S}$ can be extended to a larger set of semiabelian varieties, if necessary by extending the constant field C, $T_{\mathcal{S}}$ is a proper reduct of DCF_0. $\square$

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